THE STRATIFIED STRUCTURE OF SPACES OF SMOOTH ORBIFOLD MAPPINGS

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ABSTRACT. We consider four notions of maps between smooth C^∞ orbifolds $\mathcal{O}, \mathcal{P}$ with $\mathcal{O}$ compact (without boundary). We show that one of these notions is natural and necessary in order to uniquely define the notion of orbibundle pullback. For the notion of complete orbifold map, we show that the corresponding set of C^r maps between $\mathcal{O}$ and $\mathcal{P}$ with the C^r topology carries the structure of a smooth C^∞ Banach (r finite)/Fréchet ($r = \infty$) manifold. For the notion of complete reduced orbifold map, the corresponding set of C^r maps between $\mathcal{O}$ and $\mathcal{P}$ with the C^r topology carries the structure of a smooth C^∞ Banach (r finite)/Fréchet ($r = \infty$) orbifold. The remaining two notions carry a stratified structure: The C^r orbifold maps between $\mathcal{O}$ and $\mathcal{P}$ is locally a stratified space with strata modeled on smooth C^∞ Banach (r finite)/Fréchet ($r = \infty$) manifolds while the set of C^r reduced orbifold maps between $\mathcal{O}$ and $\mathcal{P}$ locally has the structure of a stratified space with strata modeled on smooth C^∞ Banach (r finite)/Fréchet ($r = \infty$) orbifolds. Furthermore, we give the explicit relationship between these notions of orbifold map. Applying our results to the special case of orbifold diffeomorphism groups, we show that they inherit the structure of C^∞ Banach (r finite)/Fréchet ($r = \infty$) manifolds. In fact, for r finite they are topological groups, and for $r = \infty$ they are convenient Fréchet Lie groups.

CONTENTS

1. Introduction

The purpose of this article is to provide a foundation for questions of global analysis in the category of smooth orbifolds. In the long article [12], Eells made a compelling argument that a coherent theory for problems of global analysis and non-linear functional analysis would benefit from a systematic study of the differential topology and geometry of function spaces as certain kinds of infinite dimensional manifolds. A well-known result in the theory of differentiable dynamical systems states that the set of C^r mappings $C^r(M, N)$ between C^∞ manifolds M and N with M compact has the structure of a C^∞ Banach manifold. If $r = \infty$, $C^\infty(M, N)$ becomes a C^∞ Fréchet manifold. The local model at $f \in C^r(M, N)$ is $\mathcal{D}^r(f^*(TN))$, the space of smooth sections of the pullback tangent bundle $f^*(TN)$ equipped with the C^r topology. $\mathcal{D}^r(f^*(TN))$ is a separable Banach space for $1 \leq r < \infty$ and a separable Fréchet space for $r = \infty$. For reference, see [3, 12, 15, 23, 24, 27, 28].

We wish to extend these results to the set of C^r maps from a compact smooth C^∞ orbifold $\mathcal{O}$ (without boundary) to a smooth C^∞ orbifold $\mathcal{P}$. In contrast to the manifold case, there is more than one reasonable notion of a C^r map between orbifolds. In [7], we defined a notion of (unreduced) C^r orbifold map and the notion of *reduced* orbifold map. In [8], we clarified these concepts and showed that for a compact orbifold $\mathcal{O}$ (without boundary), both the group $\mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$ of orbifold diffeomorphisms and the group $\mathrm{Diff}^r_{\mathrm{red}}(\mathcal{O})$ of reduced orbifold diffeomorphisms equipped with the C^r topology, carry the topological structure of a Banach manifold for finite r and that of a Fréchet manifold for $r = \infty$. In fact, we showed that $\mathrm{Diff}^r_{\mathrm{red}}(\mathcal{O})$ is a finite quotient of $\mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$. While our notion of orbifold map is more general than the one that typically appears in the literature, for example [2], our notion of *reduced* orbifold map agrees with that book's definition 1.3 which is the definition that appears most often.

Unfortunately, these do not exhaust the possible "reasonable" definitions of an orbifold map. To this end, we introduce two additional notions of orbifold maps, the *complete orbifold maps* and the *complete reduced orbifold maps*. Unsurprisingly, these different notions of C^r map give function space topologies that are very different, and in fact have different local models. Our results show that for the complete orbifold maps the situation is similar to the manifold case and the local model is a Banach/Fréchet space. Moreover, the other notions of orbifold map arise concretely as quotients of the complete orbifold maps and do so in a way that we are able to explicitly identify. As such quotients, the local models for the function space topologies are stratified in ways that are entirely identifiable.

While the proliferation of definitions for smooth orbifold maps may seem unnecessarily complicated, our work in [7] showed that the classical definition was deficient with respect to the reconstruction of an orbifold's structure from the algebraic structure of the corresponding diffeomorphism group. Furthermore, simple examples show that the notion of complete orbifold map is necessary to give a well-defined notion of pullback orbibundle. The need to be careful when defining pullback orbibundles was already noted in the work of Moerdijk and Pronk [26] and Chen and Ruan [10].

To help the reader who is more comfortable with the existing literature using the Lie groupoid theoretic approach to orbifolds, we make the following remarks. The Lie groupoid homomorphisms which appear in the recent orbifold literature (see, [2, 10, 11, 25, 26]) are essentially equivalent to our notion of complete reduced

orbifold map up to conjugation. In our notation, the groupoid homomorphisms are equivalent to the quotient of the complete reduced orbifold maps by the action of the lifts of the identity map. For more detail, see section 2, especially the definition of $q_\dagger$ appearing in remark 20. On the other hand, our notions of orbifold map are independent of any particular groupoid representation and thus are more natural for questions of differential topology, geometry and global analysis. We intend to elaborate on the relation between our approach and the Lie groupoid approach to orbifolds in a forthcoming article.

Lastly, we note that if M and N (as above) are, in addition, Γ-manifolds (Γ, a compact Lie group), then the space $C^r_\Gamma(M, N)$ of C^r equivariant maps from M to N is a closed C^∞ Banach submanifold of $C^r(M, N)$ [13]. In [8, Example 3.10], we observed that for a so-called good orbifold $\mathcal{O} = M/\Gamma$ (an *effective global quotient orbifold* in [2]), the orbifold diffeomorphism group $\mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$ is strictly larger than $\mathrm{Diff}^r_\Gamma(M)$, the Γ-equivariant diffeomorphism group of M. The relationship between the space of smooth orbifold maps between good orbifolds $\mathcal{O}_i = M_i/\Gamma$, and the space of equivariant maps $C^r_\Gamma(M_1, M_2)$ will be the focus of a future investigation.

We assume the reader is familiar with the notion of smooth orbifolds, and although there are many nice references for this background material such as the recently published book [2], we will use our previous work [8] as our standard reference for notation and needed definitions. We should note, however, that our definition of orbifold is modeled on the definition in Thurston [30] and that the orbifolds that concern us here are referred to as *classical effective orbifolds* in [2]. More precisely, for our definition of orbifolds, isotropy actions are always effective and we allow for singularities of codimension one. For those notions for which the existing literature is not entirely consistent, we will provide explicit definitions. Our main result is the following

Theorem 1. *Let $r \geq 1$ and let $\mathcal{O}, \mathcal{P}$ be smooth C^∞ orbifolds (without boundary) with $\mathcal{O}$ compact. Denote by $C^r_{\star\mathrm{Orb}}(\mathcal{O}, \mathcal{P})$ the set of C^r complete orbifold maps between $\mathcal{O}$ and $\mathcal{P}$ equipped with the C^r topology. Let $_\star f \in C^r_{\star\mathrm{Orb}}(\mathcal{O}, \mathcal{P})$. Then $C^r_{\star\mathrm{Orb}}(\mathcal{O}, \mathcal{P})$ is a smooth C^∞ manifold modeled locally on the topological vector space $\mathfrak{D}^r_{\mathrm{Orb}}(_\star f^*(T\mathcal{P}))$ of C^r orbisections of the pullback tangent orbibundle of $\mathcal{P}$ equipped with the C^r topology. This separable vector space is a Banach space if $1 \leq r < \infty$ and is a Fréchet space if $r = \infty$.*

Remark 2. There is no loss of generality in assuming that the orbifolds $\mathcal{O}$ and $\mathcal{P}$ are C^∞ smooth. This follows because smooth C^r orbifolds can be equipped with a compatible C^∞ structure. See [8, Proposition 3.11; 19].

As corollaries of theorem 1, we are able to prove the following structure results for our different notions of orbifold map. For the complete reduced orbifold maps we have

Corollary 3. *Let $r \geq 1$ and let $\mathcal{O}, \mathcal{P}$ be as above. Denote by $C^r_{\blacklozenge\mathrm{Orb}}(\mathcal{O}, \mathcal{P})$ the set of complete reduced C^r orbifold maps between $\mathcal{O}$ and $\mathcal{P}$ equipped with the C^r topology inherited from $C^r_{\star\mathrm{Orb}}(\mathcal{O}, \mathcal{P})$ as a quotient space. Then $C^r_{\blacklozenge\mathrm{Orb}}(\mathcal{O}, \mathcal{P})$ carries the structure of a smooth C^∞ Banach (r finite)/Fréchet (r = ∞) orbifold.*

This result essentially recovers the result of Chen [11] for r finite, where the C^r maps defined there are shown to have the structure of a smooth Banach orbifold. We have the following structure result for orbifold maps.

Corollary 4. *Let $r \geq 1$ and let $\mathcal{O}, \mathcal{P}$ be as above. Denote by $C^r_{\mathrm{Orb}}(\mathcal{O}, \mathcal{P})$ the set of C^r orbifold maps between $\mathcal{O}$ and $\mathcal{P}$ equipped with the C^r topology (as defined in [8]). Then $C^r_{\mathrm{Orb}}(\mathcal{O}, \mathcal{P})$ carries the topological structure of a stratified space with strata modeled on smooth C^∞ Banach (r finite)/Fréchet ($r = \infty$) manifolds.*

In section 5, we illustrate this phenomenon with a concrete example. Finally, for the reduced orbifold maps, we conclude

Corollary 5. *Let $r \geq 1$ and let $\mathcal{O}, \mathcal{P}$ be as above. Denote by $C^r_{\mathrm{red}}(\mathcal{O}, \mathcal{P})$ the set of C^r reduced orbifold maps between $\mathcal{O}$ and $\mathcal{P}$ equipped with the C^r topology as a quotient space. Then $C^r_{\mathrm{red}}(\mathcal{O}, \mathcal{P})$ carries the topological structure of a stratified space with strata modeled on smooth C^∞ Banach (r finite)/Fréchet ($r = \infty$) orbifolds.*

We would like to point out in each of the above results we are claiming, in part, the existence of a smooth structure modeled on Banach or Fréchet spaces. While much of the finite dimensional smooth manifold theory carries over to the Banach category, the lack of a general implicit function theorem in Fréchet spaces can cause significant difficulties [18]. In particular, there can be many inequivalent notions of differential calculus [20]. For finite order differentiability, a strong argument can be made that the Lipschitz categories Lip^r are better suited to questions of calculus than the more common C^r category. For our purposes, however, we have chosen to use the C^r category for finite order differentiability and for infinite order differentiability, we use the convenient calculus as detailed in the monographs [16, 22].

We hope that this attempt at providing a reasonably comprehensive framework for studying smooth maps between orbifolds illuminates the subtle nature of geometric and topological questions involving them and the care that must be taken in their study. The paper is divided into the following sections: Section 2 will define the four notions of orbifold map that we will be considering and how these notions are related. Section 3 defines the C^r topology on $C^r_{\star\mathrm{Orb}}(\mathcal{O}, \mathcal{P})$ with $\mathcal{O}$ compact and proves corollary 3 assuming theorem 1. Section 4 applies our results to the special case of orbifold diffeomorphisms. Section 5 provides explicit examples to show that non-orbifold structure stratifications naturally arise. Section 6 will construct the pullback orbibundle for a smooth complete orbifold map and illustrate the necessity to use complete orbifold maps in order to get a unique notion of pullback. Section 7 recalls some results about the exponential map on orbifolds and contains the proof of theorem 1. Section 8 is devoted to proofs of corollaries 4 and 5. In section 9, we collect the results of infinite-dimensional analysis that we need to substantiate our smoothness claims.

2. Four Notions of Orbifold Map

We now discuss four related definitions of maps between orbifolds. The first notion we will define is that of a complete orbifold map. It is distinguished from our previous notions of orbifold map and reduced orbifold map [8, Section 3] in that we are going to keep track of all defining data. In what follows we use the notation of [8, Section 2].

Definition 6. A C^0 *complete orbifold map* $(f, \{\tilde{f}_x\}, \{\Theta_{f,x}\})$ between C^∞ smooth orbifolds $\mathcal{O}_1$ and $\mathcal{O}_2$ consists of the following:

   (1) A continuous map $f : X_{\mathcal{O}_1} \to X_{\mathcal{O}_2}$ of the underlying topological spaces.

(2) For each $y \in \mathcal{S}_x$, a group homomorphism $\Theta_{f,y} : \Gamma_{\mathcal{S}_x} \to \Gamma_{f(y)}$.

(3) A $\Theta_{f,y}$-equivariant lift $\tilde{f}_y : \tilde{U}_y \subset \tilde{U}_{\mathcal{S}_x} \to \tilde{V}_{f(y)}$ where $(\tilde{U}_y, \Gamma_{\mathcal{S}_x}, \rho_y, \phi_y)$ is an orbifold chart at y and $(\tilde{V}_{f(y)}, \Gamma_{f(y)}, \rho_{f(y)}, \phi_{f(y)})$ is an orbifold chart at $f(y)$. That is, the following diagram commutes:

$$
\begin{array}{ccc}
\tilde{U}_y & \xrightarrow{\quad \tilde{f}_y \quad} & \tilde{V}_{f(y)} \\
\downarrow & & \downarrow \\
\tilde{U}_y/\Gamma_{\mathcal{S}_x} & \xrightarrow{\tilde{f}_y/\Theta_{f,y}(\Gamma_{\mathcal{S}_x})} & \tilde{V}_{f(y)}/\Theta_{f,y}(\Gamma_{\mathcal{S}_x}) \\
& & \downarrow \\
& & \tilde{V}_{f(y)}/\Gamma_{f(y)} \\
\downarrow & & \downarrow \\
U_y \subset U_{\mathcal{S}_x} & \xrightarrow{\quad f \quad} & V_{f(y)}
\end{array}
$$

($\star$4) (Equivalence) Two complete orbifold maps $(f, \{\tilde{f}_x\}, \{\Theta_{f,x}\})$ and $(g, \{\tilde{g}_x\}, \{\Theta_{g,x}\})$ are considered equivalent if for each $x \in \mathcal{O}_1$, $\tilde{f}_x = \tilde{g}_x$ as germs and $\Theta_{f,x} = \Theta_{g,x}$. That is, there exists an orbifold chart $(\tilde{U}_x, \Gamma_x)$ at x such that $\tilde{f}_x|_{\tilde{U}_x} = \tilde{g}_x|_{\tilde{U}_x}$ and $\Theta_{f,x} = \Theta_{g,x}$. Note that this implies that $f = g$.

Definition 7. A complete orbifold map $f : \mathcal{O}_1 \to \mathcal{O}_2$ of C^∞ smooth orbifolds is C^r *smooth* if each of the local lifts $\tilde{f}_x$ may be chosen to be C^r. Given two orbifolds $\mathcal{O}_i$, $i = 1, 2$, the set of C^r complete orbifold maps from $\mathcal{O}_1$ to $\mathcal{O}_2$ will be denoted by $C^r_{\star\mathrm{Orb}}(\mathcal{O}_1, \mathcal{O}_2)$.

If we replace ($\star$4) in definition 6 by

(4) (Equivalence) Two complete orbifold maps $(f, \{\tilde{f}_x\}, \{\Theta_{f,x}\})$ and $(g, \{\tilde{g}_x\}, \{\Theta_{g,x}\})$ are considered equivalent if for each $x \in \mathcal{O}_1$, $\tilde{f}_x = \tilde{g}_x$ as germs. That is, there exists an orbifold chart $(\tilde{U}_x, \Gamma_x)$ at x such that $\tilde{f}_x|_{\tilde{U}_x} = \tilde{g}_x|_{\tilde{U}_x}$ (which as before implies $f = g$),

where we have dropped the requirement that $\Theta_{f,x} = \Theta_{g,x}$, we recover the notion of *orbifold map* $(f, \{\tilde{f}_x\})$ which appeared in [8, Section 3]. Thus, the set of orbifold maps $C^r_{\mathrm{Orb}}(\mathcal{O}_1, \mathcal{O}_2)$ can be regarded as the equivalence classes of complete orbifold maps under the less restrictive set-theoretic equivalence (4). The following simple example is illustrative.

Example 8. Let $\mathcal{O}$ be the orbifold $\mathbb{R}/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts on $\mathbb{R}$ via $x \to -x$ and $f : \mathcal{O} \to \mathcal{O}$ is the constant map $f \equiv 0$. The underlying topological space $X_\mathcal{O}$ of $\mathcal{O}$ is $[0, \infty)$ and the isotropy subgroups are trivial for $x \in (0, \infty)$ and $\mathbb{Z}_2$ for $x = 0$. The map $\tilde{f}_0 \equiv 0$ is a local equivariant lift of f at $x = 0$ using either of the homomorphisms $\Theta_{f,0} = \mathrm{Id}$ or $\Theta'_{f,0} \equiv e$. Of course, for $x \neq 0$, we set $\tilde{f}_x \equiv 0$ and $\Theta_{f,x} = \Theta'_{f,x} = $ the trivial homomorphism $\Gamma_x = e \mapsto e \in \Gamma_0 = \mathbb{Z}_2$. Thus, as *complete* orbifold maps $(f, \{\tilde{f}_x\}, \{\Theta_{f,x}\}) \neq (f, \{\tilde{f}_x\}, \{\Theta'_{f,x}\})$. However, simply as orbifold maps, they are considered equal.

If we replace ($\star$4) in definition 6 by

($\blacklozenge$4) (Equivalence) Two complete orbifold maps $(f, \{\tilde{f}_x\}, \{\Theta_{f,x}\})$ and $(g, \{\tilde{g}_x\}, \{\Theta_{g,x}\})$ are considered equivalent if $f = g$ and for each $x \in \mathcal{O}_1$, we have $\Theta_{f,x} = \Theta_{g,x}$,

where we have dropped the requirement that the germs of the lifts $\tilde{f}_x$ and $\tilde{g}_x$ agree, we obtain a new notion of orbifold map $(f, \{\Theta_{f,x}\})$ which we call a *complete reduced orbifold map.* The set of smooth complete reduced orbifold maps will be denoted by $C^r_{\blacklozenge\mathrm{Orb}}(\mathcal{O}_1, \mathcal{O}_2)$. As before, it is clear that $C^r_{\blacklozenge\mathrm{Orb}}(\mathcal{O}_1, \mathcal{O}_2)$ is a set-theoretic quotient of $C^r_{\star\mathrm{Orb}}(\mathcal{O}_1, \mathcal{O}_2)$.

If we replace (4) in the definition of orbifold map, or ($\blacklozenge$4) in the definition of complete reduced orbifold map, by

($\bullet$4) (Equivalence) Two orbifold maps $(f, \{\tilde{f}_x\})$ and $(g, \{\tilde{g}_x\})$, (or, complete reduced orbifold maps $(f, \{\Theta_{f,x}\})$ and $(g, \{\Theta_{g,x}\})$) are considered equivalent if $f = g$.

we obtain the notion of *reduced orbifold map* from [7]. The set of smooth reduced orbifold maps will be denoted by $C^r_{\mathrm{red}}(\mathcal{O}_1, \mathcal{O}_2)$. Like before, it is clear that $C^r_{\mathrm{red}}(\mathcal{O}_1, \mathcal{O}_2)$ is a set-theoretic quotient of both $C^r_{\mathrm{Orb}}(\mathcal{O}_1, \mathcal{O}_2)$ and $C^r_{\blacklozenge\mathrm{Orb}}(\mathcal{O}_1, \mathcal{O}_2)$.

Notation. Since we will often need to distinguish between these various notions of orbifold maps, we will denote a complete orbifold map $(f, \{\tilde{f}_x\}, \{\Theta_{f,x}\})$ by $_\star f$, and represent an orbifold map $(f, \{\tilde{f}_x\})$ simply by f as in [8], a complete reduced orbifold map $(f, \{\Theta_{f,x}\})$ by $_\blacklozenge f$, and a reduced orbifold map by $_\bullet f$.

Diagrammatically, we have the following:

where the q's represent the respective set-theoretic quotient maps. Understanding how these notions are related in the special case of the identity map is crucial in what follows.

Example 9. (Lifts of the Identity Map) Consider the identity map $\mathrm{Id} : \mathcal{O} \to \mathcal{O}$. Let $x \in \mathcal{O}$ and $(\tilde{U}_x, \Gamma_x)$ be an orbifold chart at x. From the definition of orbifold map, it follows (since Γ_x is finite) that there exists $\gamma \in \Gamma_x$ such that a lift $\widetilde{\mathrm{Id}}_x : \tilde{U}_x \to \tilde{U}_x$ is given by $\widetilde{\mathrm{Id}}_x(\tilde{y}) = \gamma \cdot \tilde{y}$ for all $\tilde{y} \in \tilde{U}_x$. Since $\widetilde{\mathrm{Id}}_x$ is $\Theta_{\mathrm{Id},x}$ equivariant we have for $\delta \in \Gamma_x$:

$$\widetilde{\mathrm{Id}}_x(\delta \cdot \tilde{y}) = \Theta_{\mathrm{Id},x}(\delta) \cdot \widetilde{\mathrm{Id}}_x(\tilde{y}) \quad \text{hence}$$
$$\gamma\delta \cdot \tilde{y} = \Theta_{\mathrm{Id},x}(\delta)\gamma \cdot \tilde{y} \quad \text{which implies}$$
$$\text{since } \Gamma_x \text{ acts effectively that}$$
$$\gamma\delta = \Theta_{\mathrm{Id},x}(\delta)\gamma \quad \text{or, equivalently,}$$
$$\Theta_{\mathrm{Id},x}(\delta) = \gamma\delta\gamma^{-1}$$

Thus, the isomorphism $\Theta_{\mathrm{Id},x}$ is completely determined by the choice of local lift $\widetilde{\mathrm{Id}}_x$. This implies that the group $\mathcal{ID}$ of orbifold maps covering the identity may be regarded as the same as the group $_\star\mathcal{ID}$ of complete orbifold maps covering the identity. That is, we have the bijective correspondence

$$(\mathrm{Id}, \{\tilde{y} \mapsto \gamma_x \cdot \tilde{y}\}, \{\Theta_{\mathrm{Id},x}\}) \longleftrightarrow (\mathrm{Id}, \{\tilde{y} \mapsto \gamma_x \cdot \tilde{y}\}).$$

Suppose now that $\{U_{x_i}\}$ is a countable (possibly finite) cover of $\mathcal{O}$ by charts. Then $\mathcal{ID}$ can be regarded as a subgroup of the product $\prod \Gamma_{x_i}$ as in the proof of corollary 1.2 in [8]. Two inner automorphisms, $\delta \mapsto \gamma_i \delta \gamma_i^{-1}$, give rise to the same automorphism of Γ_x precisely when $\gamma_1 = \zeta \gamma_2$ where $\zeta \in C(\Gamma_x)$, the center of Γ_x. Thus, if we let $C = C(\mathcal{ID}) \subset \prod C(\Gamma_{x_i})$, then one can see that the complete reduced lifts of the identity $_\bullet\mathcal{ID} \cong {}_\star\mathcal{ID}/C$, where the free C-action on $_\star\mathcal{ID}$ is defined by

$$(\zeta_i) \cdot (\mathrm{Id}, \{\tilde{y} \mapsto \gamma_{x_i} \cdot \tilde{y}\}, \{\Theta_{\mathrm{Id},x_i}\}) = (\mathrm{Id}, \{\tilde{y} \mapsto (\zeta_i \gamma_{x_i}) \cdot \tilde{y}\}, \{\zeta_i \Theta_{\mathrm{Id},x_i} \zeta_i^{-1} = \Theta_{\mathrm{Id},x_i}\}).$$

Also, note that the correspondence $_\star\mathcal{ID} \leftrightarrow \mathcal{ID}$ gives an isomorphism $_\bullet\mathcal{ID} \cong \mathcal{ID}/C$ which in turn is isomorphic to $\mathrm{Inn}(\mathcal{ID})$, the group of inner automorphisms of $\mathcal{ID}$. Thus, we have the exact sequence

$$1 \to C(\mathcal{ID}) \to {}_\star\mathcal{ID} = \mathcal{ID} \to {}_\bullet\mathcal{ID} \to 1$$

Notation. For a (not necessarily compact) orbifold $\mathcal{N}$, we will use the notation $\mathcal{ID}_{\mathcal{N}}$ to denote the group of orbifold lifts of the identity map $\mathrm{Id} : \mathcal{N} \to \mathcal{N}$. Suppose $f : \mathcal{O}_1 \to \mathcal{O}_2$ and let the orbifold $\mathcal{N}$ be an open neighborhood of the image $f(\mathcal{O}_1)$. For an orbifold map $_{\{\cdot\}}f$ (of any type) and $I = (\mathrm{Id}, \{\eta_x \cdot \tilde{y}\}) \in \mathcal{ID}_{\mathcal{N}}$ we can compute $I \circ {}_{\{\cdot\}}f$. Namely,

$$I \circ {}_\star f = I \circ (f, \{\tilde{f}_x\}, \{\Theta_{f,x}\}) = (f, \{\eta_x \cdot \tilde{f}_x\}, \{\gamma \mapsto \eta_x \Theta_{f,x}(\gamma) \eta_x^{-1}\})$$
$$I \circ {}_\bullet f = I \circ (f, \{\Theta_{f,x}\}) = (f, \{\gamma \mapsto \eta_x \Theta_{f,x}(\gamma) \eta_x^{-1}\})$$
$$I \circ f = I \circ (f, \{\tilde{f}_x\}) = (f, \{\eta_x \cdot \tilde{f}_x\})$$
$$I \circ {}_\bullet f = {}_\bullet f$$

Suppose $\{\Gamma_x\}$ denotes the family of isotropy groups for an orbifold $\mathcal{N}$ and for subgroups $\Lambda_x \subset \Gamma_x$, let $\{\Lambda_x\}$ denote the corresponding family of subgroups. In what follows, we will use the notation $(\mathcal{ID}_{\mathcal{N}})_{\{\Lambda_x\}}$ for the subgroup of $\mathcal{ID}_{\mathcal{N}}$ defined by

$$\{I \in \mathcal{ID}_{\mathcal{N}} \mid I = (\mathrm{Id}, \{\tilde{y} \mapsto \lambda_x \cdot \tilde{y}\}) \text{ where } \lambda_x \in \Lambda_x \text{ for all } x\}.$$

Lastly, for a fixed orbifold map $_{\{\cdot\}}f$ (of any type), we let $(\mathcal{ID}_{\mathcal{N}}) \cdot {}_{\{\cdot\}}f$ denote the orbit under the action of $\mathcal{ID}_{\mathcal{N}}$:

$$(\mathcal{ID}_{\mathcal{N}}) \cdot {}_{\{\cdot\}}f = \{I \circ {}_{\{\cdot\}}f \mid I \in \mathcal{ID}_{\mathcal{N}}\}$$

and we let $(\mathcal{ID}_{\mathcal{N}})_{{}_{\{\cdot\}}f}$ denote the corresponding isotropy subgroup of $_{\{\cdot\}}f$ under the action of $\mathcal{ID}_{\mathcal{N}}$:

$$\{I \in \mathcal{ID}_{\mathcal{N}} \mid I \circ {}_{\{\cdot\}}f = {}_{\{\cdot\}}f\}.$$

It is also important to note that $\mathcal{ID}_{\mathcal{N}}$ is a finite group in the special case that the source orbifold $\mathcal{O}_1$ is compact: one may choose the open neighborhood $\mathcal{N}$ of $f(\mathcal{O}_1)$ to be relatively compact and since $\mathcal{N}$ can be covered by finitely many orbifold charts $\{U_{x_i}\}$, the observation that $\mathcal{ID} \subset \prod \Gamma_{x_i}$ from example 9 is enough to show that, in this case, $\mathcal{ID}_{\mathcal{N}}$ is finite.

Implications for the definition of orbifold structure. Recall the following commutative diagram of maps which appears in the definition of a smooth classical effective orbifold [8]:

$$
\begin{array}{ccc}
\tilde{U}_z & \xrightarrow{\quad\tilde{\psi}_{zx}\quad} & \tilde{U}_x \\
\downarrow & & \downarrow \\
U_z \cong \tilde{U}_z/\Gamma_z & \xrightarrow{\quad\psi_{zx}\quad} & \tilde{U}_x/\Gamma_x \cong U_x
\end{array}
$$

where for a neighborhood $U_z \subset U_x$ with corresponding $\tilde{U}_z$, and isotropy group Γ_z, there is an open embedding $\tilde{\psi}_{zx} : \tilde{U}_z \to \tilde{U}_x$ covering the inclusion $\psi_{zx} : U_z \hookrightarrow U_x$ and an injective homomorphism $\theta_{zx} : \Gamma_z \to \Gamma_x$ so that $\tilde{\psi}_{zx}$ is equivariant with respect to θ_{zx}. For the standard definition of orbifold which appears in the literature, it is understood that $\tilde{\psi}_{zx}$ is defined only up to composition with elements of Γ_x, and θ_{zx} defined only up to conjugation by elements of Γ_x. However, here, we may regard ψ_{zx} as being from any of the notions of orbifold map we have defined, thus giving an orbifold $\mathcal{O}$, or more precisely, an orbifold atlas for $\mathcal{O}$, one of four different structures depending on how one keeps track of lifts ψ_{zx} and homomorphisms θ_{zx}. Thus, it makes sense to speak of a *complete orbifold structure* $_\star\mathcal{O}$, a *complete reduced orbifold structure* $_\blacklozenge\mathcal{O}$, an *orbifold structure* $\mathcal{O}$, and lastly, a *reduced orbifold structure* $_\bullet\mathcal{O}$. Thus, the standard definition of orbifold would correspond to our notion of a reduced orbifold structure. The reader should take care to note that the term *reduced orbifold* also has been used in the study of so-called noneffective orbifolds [10]. Our use of the term *reduced orbifold structure* is unrelated to this. In this paper, the term orbifold will require that the chart maps ψ_{zx} be regarded as orbifold maps in $C^r_{\mathrm{Orb}}(U_z, U_x)$ as defined above. We also point out that there is no fundamental difference between a complete orbifold structure $_\star\mathcal{O}$ and an orbifold structure $\mathcal{O}$ and that any reduced orbifold structure $_\bullet\mathcal{O}$ is obtained as a quotient an orbifold structure $\mathcal{O}$ by the action of $\mathcal{ID}$ on orbifold atlases. This follows from example 9 and the fact that any two lifts of ψ_{zx} must differ by a lift of the identity map on U_x. Lastly, we remark that, in general, for an orbifold structure $\mathcal{O}$, $\tilde{\psi}_{zx} \neq \tilde{\psi}_{yx} \circ \tilde{\psi}_{zy}$ when $U_z \subset U_y \subset U_x$, but there will be an element $\delta \in \Gamma_x$ such that $\delta \cdot \tilde{\psi}_{zx} = \tilde{\psi}_{yx} \circ \tilde{\psi}_{zy}$ and $\delta \cdot \theta_{zx}(\gamma) \cdot \delta^{-1} = \theta_{yx} \circ \theta_{zy}(\gamma)$.

Relationship among the different notions of orbifold map. In this subsection we give a series of lemmas that discuss the relationship among the various notions of orbifold map for a fixed map $f : \mathcal{O}_1 \to \mathcal{O}_2$. In section 3, we will topologize these sets of mappings and discuss the local structure of these relationships. Our first lemma makes explicit the relationship between the complete reduced orbifold maps and the complete orbifold maps.

Lemma 10. *Let* $_\star f, _\star f' \in C^r_{\star\mathrm{Orb}}(\mathcal{O}_1, \mathcal{O}_2)$ *be complete orbifold maps which represent the same complete reduced orbifold map. That is,* $_\blacklozenge f = _\blacklozenge f' \in C^r_{\blacklozenge\mathrm{Orb}}(\mathcal{O}_1, \mathcal{O}_2)$, *so that* $_\star f = (f, \{\tilde{f}_x\}, \{\Theta_{f,x}\})$ *and* $_\star f' = (f, \{\tilde{f}'_x\}, \{\Theta_{f,x}\})$. *Let* $C_x = C_{\Gamma_{f(x)}}(\Theta_{f,x}(\Gamma_x))$ *denote the centralizer of* $\Theta_{f,x}(\Gamma_x)$ *in* $\Gamma_{f(x)}$. *Then there is an orbifold* $\mathcal{N}$ *which is an open neighborhood of* $f(\mathcal{O}_1)$ *in* $\mathcal{O}_2$ *and an orbifold map* $I \in (\mathcal{ID}_\mathcal{N})_{\{C_x\}}$ *such that* $_\star f = I \circ _\star f'$. *Moreover, if the stated condition holds for two complete orbifold maps* $_\star f$ *and* $_\star f'$, *then* $_\blacklozenge f = _\blacklozenge f'$.

Proof. Since $\tilde{f}_x$ and $\tilde{f}'_x$ are local lifts of the same map f, there exists $\eta_x \in \Gamma_{f(x)}$ such that $\tilde{f}'_x(\tilde{y}) = \eta_x \cdot \tilde{f}_x(\tilde{y})$ for all $\tilde{y} \in \tilde{U}_x$. Thus, for all $\gamma \in \Gamma_x$ we have, on one hand, the equivariance relation $\tilde{f}'_x(\gamma \cdot \tilde{y}) = \Theta_{f,x}(\gamma) \cdot \tilde{f}'_x(\tilde{y})$ while on the other hand, the equivariance relation must be $\tilde{f}'_x(\gamma \cdot \tilde{y}) = \eta_x \cdot \tilde{f}_x(\gamma \cdot \tilde{y}) = \eta_x \Theta_{f,x}(\gamma) \cdot \tilde{f}_x(\tilde{y}) = \eta_x \Theta_{f,x}(\gamma)\eta_x^{-1} \cdot \tilde{f}'_x(\tilde{y})$. This implies that $\Theta_{f,x}(\gamma) = \eta_x \Theta_{f,x}(\gamma)\eta_x^{-1}$ and thus $\eta_x \in C_x$. The orbifold $\mathcal{N}$ may be taken to be $\mathcal{N} = \cup_{x \in \mathcal{O}_1} V_{f(x)}$, where $V_{f(x)}$ is an orbifold chart about $f(x) \in \mathcal{O}_2$. We have thus shown the first statement of the lemma, and the last statement is clear from our computation above and the definitions. $\square$

Example 11. Let $\mathcal{O}$ be as in example 8. Consider the complete orbifold map $_\star f = (f, \{\tilde{f}_x\}, \{\Theta_{f,x}\})$ which covers the inclusion map $f : \mathcal{O} \to \mathcal{O} \times \mathcal{O} \times \mathcal{O}$, $y \mapsto (y, 0, 0)$, where $\tilde{f}_x(\tilde{y}) = (\tilde{y}, 0, 0)$ and $\Theta_{f,0}(\gamma) = (\gamma, e, e) \in \Gamma_{(0,0,0)} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ (for $x \neq 0$, $\Theta_{f,x}$ is the trivial homomorphism since $\Gamma_x = \{e\}$). Now, $\eta_0 = (e, \gamma, e) \in C_{\Gamma_{(0,0,0)}}(\Theta_{f,0}(\Gamma_0))$ and $\eta_0 \cdot \tilde{f}_0(\tilde{y}) = \eta_0 \cdot (\tilde{y}, 0, 0) = (\tilde{y}, 0, 0) = \tilde{f}_0(\tilde{y})$. Now let the (finite) group $(\mathbb{JD}_\mathcal{N})_{\{C_x\}}$ be as in lemma 10. For fixed $_\star f$, let $(\mathbb{JD}_\mathcal{N})_{\{C_x\}} \cdot {}_\star f$ denote the orbit of $_\star f$. This example shows that the orbit map $(\mathrm{Id}, \{\lambda_x \cdot \tilde{z}\}) \mapsto (\mathrm{Id}, \{\lambda_x \cdot \tilde{z}\}) \circ {}_\star f$ may have nontrivial, (but finite) isotropy. Thus, if we let $(\mathbb{JD}_\mathcal{N})_{\star f} \subset (\mathbb{JD}_\mathcal{N})_{\{C_x\}}$ denote the isotropy subgroup of $_\star f$, then $(\mathbb{JD}_\mathcal{N})_{\{C_x\}}/(\mathbb{JD}_\mathcal{N})_{\star f} \cong (\mathbb{JD}_\mathcal{N})_{\{C_x\}} \cdot {}_\star f$ is a homeomorphism (of discrete sets).

The next lemma describes the relationship between the orbifold maps and the reduced orbifold maps.

Lemma 12. *Let $f, f' \in C_{\mathrm{Orb}}^r(\mathcal{O}_1, \mathcal{O}_2)$ be orbifold maps which represent the same reduced orbifold map. That is, $_\bullet f = {}_\bullet f' \in C_{\mathrm{red}}^r(\mathcal{O}_1, \mathcal{O}_2)$, so that $f = (f, \{\tilde{f}_x\})$ and $f' = (f, \{\tilde{f}'_x\})$. Then there is an orbifold $\mathcal{N}$ which is an open neighborhood of $f(\mathcal{O}_1)$ in $\mathcal{O}_2$ and an orbifold map $I = (\mathrm{Id}, \{\eta_x \cdot \tilde{z}\}) \in \mathbb{JD}_\mathcal{N}$ with $\eta_x \in \Gamma_{f(x)}$ such that $f = I \circ f'$. Moreover, if the stated condition holds for two orbifold maps f and f', then $_\bullet f = {}_\bullet f'$.*

Proof. $\mathcal{N}$ can be chosen as in lemma 10, and the proof follows from corollary 1.2 in [8]. $\square$

Remark 13. Similar to the situation described in example 11, example 8 shows that the orbit map $\mathbb{JD}_\mathcal{N} \cdot f$ may have nontrivial isotropy.

Next, we describe the relationship between the complete orbifold maps and the orbifold maps.

Lemma 14. *Let $_\star f, {}_\star f' \in C_{\star\mathrm{Orb}}^r(\mathcal{O}_1, \mathcal{O}_2)$ be complete orbifold maps which represent the same orbifold map. That is, $f = f' \in C_{\mathrm{Orb}}^r(\mathcal{O}_1, \mathcal{O}_2)$, so that $_\star f = (f, \{\tilde{f}_x\}, \{\Theta_{f,x}\})$ and $_\star f' = (f, \{\tilde{f}_x\}, \{\Theta'_{f,x}\})$. Then, for each $x \in \mathcal{O}_1$, $\gamma \in \Gamma_x$, and $\tilde{y} \in \tilde{U}_x$ we have*

$$\left\{ \left[\Theta'_{f,x}(\gamma)\right]^{-1} \left[\Theta_{f,x}(\gamma)\right] \right\} \cdot \tilde{f}_x(\tilde{y}) = \tilde{f}_x(\tilde{y}).$$

Moreover, if the stated condition holds for two complete orbifold maps $_\star f$ and $_\star f'$, then $f = f'$.

Proof. For all $\gamma \in \Gamma_x$ and $\tilde{y} \in \tilde{U}_x$, we have, $\Theta_{f,x}(\gamma) \cdot \tilde{f}_x(\tilde{y}) = \tilde{f}_x(\gamma \cdot \tilde{y}) = \Theta'_{f,x}(\gamma) \cdot \tilde{f}_x(\tilde{y})$ and the first statement follows. To see the last statement, let $_\star f = (f, \{\tilde{f}_x\}, \{\Theta_{f,x}\})$ and $_\star f' = (f, \{\tilde{f}'_x\}, \{\Theta'_{f,x}\})$. Note that the condition stated implies that $\tilde{f}_x(\tilde{y}) = \Theta_{f,x}(e) \cdot \tilde{f}_x(\tilde{y}) = \Theta'_{f,x}(e) \cdot \tilde{f}'_x(\tilde{y}) = \tilde{f}'_x(\tilde{y})$. $\square$

Remark 15. Notice that this relationship is qualitatively different than the relationships described in lemmas 10, 12 and 17, in that it is given as an equality of *actions* of $\Theta_{f,x}(\Gamma_x)$, $\Theta'_{f,x}(\Gamma_x)$ on the image $\tilde{f}_x(\tilde{U}_x)$ and not as an equality of $\Theta_{f,x}$ and $\Theta'_{f,x}$ as homomorphisms themselves. That is, the representation of $\Theta_{f,x}(\Gamma_x)$ and $\Theta'_{f,x}(\Gamma_x)$ induce actions that when restricted to $\tilde{f}_x(\tilde{U}_x)$ are equal.

Remark 16. Example 8 exhibits the behavior described in lemma 14. A slightly less trivial example is to consider the inclusion map of example 11: $f : \mathcal{O} \to \mathcal{O} \times \mathcal{O} \times \mathcal{O}$, $y \mapsto (y, 0, 0)$, where $\tilde{f}_x(\tilde{y}) = (\tilde{y}, 0, 0)$. Note that $\tilde{f}_0$ is equivariant with respect to both $\Theta_{f,0}(\gamma) = (\gamma, e, e)$ and $\Theta'_{f,0}(\gamma) = (\gamma, \gamma, \gamma)$.

The next two lemmas describe the relationship between the complete reduced orbifold maps and the reduced orbifold maps. Given the conclusion of corollary 5, this relationship is necessarily more complicated.

Lemma 17. *Let* $_\bullet f \in C^r_{\bullet\mathrm{Orb}}(\mathcal{O}_1, \mathcal{O}_2)$ *be a complete reduced orbifold map and let* $\mathcal{N}$ *be an open neighborhood of* $f(\mathcal{O}_1)$ *in* $\mathcal{O}_2$. *Let* $I = (\mathrm{Id}, \{\eta_x \cdot \tilde{z}\}) \in \mathcal{ID}_{\mathcal{N}}$ *with* $\eta_x \in \Gamma_{f(x)}$. *If the complete reduced orbifold map* $_\bullet f' = I \circ {}_\bullet f$, *then* $_\bullet f = {}_\bullet f' \in C^r_{\mathrm{red}}(\mathcal{O}_1, \mathcal{O}_2)$. *Furthermore, if* $_\bullet f'$ *is in the orbit* $(\mathcal{ID}_{\mathcal{N}}) \cdot {}_\bullet f$, *then* $\Theta'_{f,x}(\gamma) = \eta_x \Theta_{f,x}(\gamma)\eta_x^{-1}$.

Proof. Let $_\bullet f = (f, \{\Theta_{f,x}\})$ and $_\bullet f' = (f', \{\Theta'_{f,x}\})$. Let $\tilde{f}_x, \tilde{f}'_x$ be local lifts equivariant with respect to $\Theta_{f,x}, \Theta'_{f,x}$ respectively. Since $_\bullet f' = I \circ {}_\bullet f$, $\tilde{f}'_x(\tilde{y}) = \eta_x \cdot \tilde{f}_x(\tilde{y})$ for all $\tilde{y} \in \tilde{U}_x$, so $\tilde{f}_x$ and $\tilde{f}'_x$ are local lifts of the same map $f = f'$. This implies $_\bullet f = {}_\bullet f'$. The last statement follows from the way $\mathcal{ID}_{\mathcal{N}}$ acts on $_\bullet f$. □

Remark 18. Here, like before, the orbit map $(\mathrm{Id}, \{\eta_x \cdot \tilde{z}\}) \mapsto (\mathrm{Id}, \{\eta_x \cdot \tilde{z}\}) \circ {}_\bullet f$ may have nontrivial, (but finite) isotropy. In fact, $(\mathcal{ID}_{\mathcal{N}})_{\bullet f} = (\mathcal{ID}_{\mathcal{N}})_{\{C_x\}}$, the orbifold map lifts of the identity given by elements of $C_{\Gamma_{f(x)}}(\Theta_{f,x}(\Gamma_x))$ described in lemma 10.

In light of lemma 17, we define an equivalence relation the preimage $q_\blacktriangledown^{-1}(_\bullet f)$:

$$(\dagger) \qquad {}_\bullet f = (f, \{\Theta_{f,x}\}) \sim {}_\bullet f' = (f', \{\Theta'_{f,x}\}) \Longleftrightarrow {}_\bullet f' = I \circ {}_\bullet f$$

for some $I \in \mathcal{ID}_{\mathcal{N}}$. That is, for all $x \in \mathcal{O}_1$, there exists $\eta_x \in \Gamma_{f(x)}$ such that $\Theta'_{f,x}(\gamma) = \eta_x \Theta_{f,x}(\gamma)\eta_x^{-1}$ for all $\gamma \in \Gamma_x$. Denote the equivalence class of $_\bullet f$ by $[_\bullet f]$.

Lemma 19. *Let* $[_\bullet f = (f, \{\Theta_{f,x}\})] \neq [_\bullet f' = (f, \{\Theta'_{f,x}\})]$ *be different equivalence classes of complete reduced orbifold maps which represent the same reduced orbifold map. That is,* $_\bullet f = {}_\bullet f' \in C^r_{\mathrm{red}}(\mathcal{O}_1, \mathcal{O}_2)$. *Then there exist local lifts* $\{\tilde{f}_x\}$ *which are equivariant with respect to both* $\{\Theta_{f,x}\}$ *and* $\{\eta_x \Theta'_{f,x} \eta_x^{-1}\}$ *with* $\{\Theta_{f,x}\} \neq \{\eta_x \Theta'_{f,x} \eta_x^{-1}\}$ *as homomorphisms. However, for each* $x \in \mathcal{O}_1$, $\gamma \in \Gamma_x$, *and* $\tilde{y} \in \tilde{U}_x$ *we have as actions on* $\tilde{f}_x(\tilde{U}_x)$

$$(\ddagger) \qquad \Theta_{f,x}(\gamma) \cdot \tilde{f}_x(\tilde{y}) = \eta_x \Theta'_{f,x}(\gamma)\eta_x^{-1} \cdot \tilde{f}_x(\tilde{y}).$$

Proof. Since $_\bullet f = {}_\bullet f'$, there exists $\eta_x \in \Gamma_{f(x)}$ such that $\tilde{f}_x(\tilde{y}) = \eta_x \cdot \tilde{f}'_x(\tilde{y})$ for all $\tilde{y} \in \tilde{U}_x$. Thus, we conclude that $\tilde{f}_x = \eta_x \cdot \tilde{f}'_x$ is also equivariant with respect to $\eta_x \Theta'_{f,x} \eta_x^{-1}$. Since $[_\bullet f] \neq [_\bullet f']$, we have $\{\Theta_{f,x}\} \neq \{\eta_x \Theta'_{f,x} \eta_x^{-1}\}$ as homomorphisms. □

Remark 20. Example 8 illustrates the phenomena dealt with in lemma 19. Lemmas 17 and 19 show that the quotient map $q_{\blacktriangledown} : C^r_{\blacklozenge\mathrm{Orb}}(\mathcal{O}_1, \mathcal{O}_2) \to C^r_{\mathrm{red}}(\mathcal{O}_1, \mathcal{O}_2)$ factors $q_{\blacktriangledown} = q_{\ddagger} \circ q_{\dagger}$:

$$\blacklozenge f \xrightarrow{\ q_{\dagger}\ } [\blacklozenge f] \xrightarrow{\ q_{\ddagger}\ } \bullet f$$
$$\underset{q_{\blacktriangledown}}{\longrightarrow}$$

where $q_{\dagger}$, $q_{\ddagger}$ represent the quotient maps under the equivalences (†) and (‡), respectively.

3. Function Space Topologies

It is easy to define a C^s topology ($1 \leq s \leq r$) on the set of smooth complete orbifold maps $C^r_{\star\mathrm{Orb}}(\mathcal{O}, \mathcal{P})$ with $\mathcal{O}$ compact. Although much of what we do applies to noncompact $\mathcal{O}$ we will assume $\mathcal{O}$ to be compact. As such, implicit in some of the discussion is that $\mathcal{O}$ has been equipped with a finite covering by orbifold charts. The topologies we define have already been shown to be independent of these choices of charts [8].

Definition 21. Let $_\star f = (f, \{\tilde{f}_x\}, \{\Theta_{f,x}\})$, $_\star g = (g, \{\tilde{g}_x\}, \{\Theta_{g,x}\}) \in C^r_{\star\mathrm{Orb}}(\mathcal{O}, \mathcal{P})$. Then a C^s neighborhood of $_\star f$ is defined to be

$$\mathcal{N}^s(_\star f, \varepsilon) = \{_\star g = (g, \{\tilde{g}_x\}, \{\Theta_{g,x}\}) \mid$$
$$g \in \mathcal{N}^s(f, \varepsilon) \text{ and } \theta_{f(x)z} \circ \Theta_{f,x} = \theta_{g(x)z} \circ \Theta_{g,x} \text{ for all } x \in \mathcal{O}\}$$

where $\mathcal{N}^s(f, \varepsilon)$ is the C^s orbifold map neighborhood of f defined in [8]. $\theta_{f(x)z} \circ \Theta_{f,x} = \theta_{g(x)z} \circ \Theta_{g,x}$ is to be interpreted as follows: There is a small enough orbifold chart $\tilde{U}_x$ about x, such that the images of both $\tilde{f}_x(\tilde{U}_x)$ and $\tilde{g}_x(\tilde{U}_x)$ are contained in a single orbifold chart $\tilde{V}_z$ and $\theta_{f(x)z} \circ \Theta_{f,x} = \theta_{g(x)z} \circ \Theta_{g,x}$ where $\theta_{f(x)z}, \theta_{g(x)z} : \Gamma_{f(x)}, \Gamma_{g(x)} \hookrightarrow \Gamma_z$ are the injective homomorphisms given in the definition of orbifold. It is important to note that this condition is more than just an isomorphism of groups, but is an equality of their representations as actions on $\tilde{V}_z$. The collection of sets of this type form a subbasis for the corresponding C^s topology on $C^r_{\star\mathrm{Orb}}(\mathcal{O}, \mathcal{P})$.

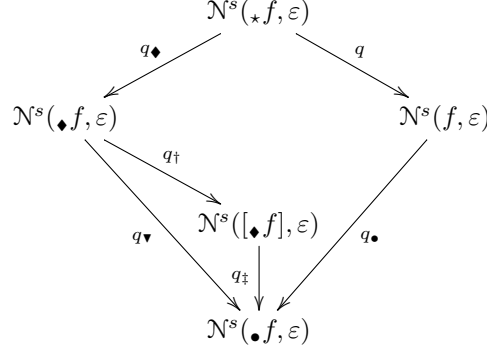Similarly, a C^s neighborhood of $_\blacklozenge f = (f, \{\Theta_{f,x}\})$ is defined to be

$$\mathcal{N}^s(_\blacklozenge f, \varepsilon) = \{_\blacklozenge g = (g, \{\Theta_{g,x}\}) \in C^r_{\blacklozenge\mathrm{Orb}}(\mathcal{O}, \mathcal{P}) \mid$$
$$_\bullet g \in \mathcal{N}^s(_\bullet f, \varepsilon) \text{ and } \theta_{f(x)z} \circ \Theta_{f,x} = \theta_{g(x)z} \circ \Theta_{g,x} \text{ for all } x \in \mathcal{O}\}$$

where $\mathcal{N}^s(_\bullet f, \varepsilon) = \{_\bullet g \in C^r_{\mathrm{red}}(\mathcal{O}, \mathcal{P}) \mid g \in \mathcal{N}^s(I \circ f, \varepsilon) \text{ for some } I \in \mathcal{ID}_\mathcal{N}\}$. Here, $\mathcal{N}$ denotes, as usual, an open neighborhood of the image $f(\mathcal{O})$.

Observation. Suppose $_\star f = (f, \{\tilde{f}_x\}, \{\Theta_{f,x}\})$ and $_\star f' = (f, \{\tilde{f}_x\}, \{\Theta'_{f,x}\})$ are two complete orbifold maps in $C^r_{\star\mathrm{Orb}}(\mathcal{O}, \mathcal{P})$ such that $f = f'$ as orbifold maps. Then $_\star f' = (f, \{\tilde{f}_x\}, \{\Theta'_{f,x}\}) \notin \mathcal{N}^s(_\star f, \varepsilon)$ for any ε unless $\{\Theta'_{f,x}\} = \{\Theta_{f,x}\}$. Otherwise, it would follow that $\theta_{f(x)z} \circ \Theta_{f,x} = \theta_{f(x)z} \circ \Theta'_{f,x}$, contradicting injectivity of $\theta_{f(x)z}$. Of course the same argument shows that if $g = g'$ as orbifold maps, then $_\star g = (g, \{\tilde{g}_x\}, \{\Theta_{g,x}\})$ and $_\star g' = (g, \{\tilde{g}_x\}, \{\Theta'_{g,x}\})$ cannot both belong to a neighborhood $\mathcal{N}^s(_\star f, \varepsilon)$ unless $\{\Theta_{g,x}\} = \{\Theta'_{g,x}\}$. As a consequence, we see that the preimage $q^{-1}(\mathcal{N}^s(f, \varepsilon)) \subset C^r_{\star\mathrm{Orb}}(\mathcal{O}, \mathcal{P})$ is a (finite) disjoint union of neighborhoods of the form $\mathcal{N}^s(_\star f_i, \varepsilon)$ where $_\star f_i = (f, \{\tilde{f}_x\}, \{\Theta_{f,x}\}_i)$. Similarly, we see that the preimage

$q_{\blacktriangledown}^{-1}(\mathcal{N}^s({}_{\bullet}f, \varepsilon)) \subset C_{\blacklozenge\mathrm{Orb}}^r(\mathcal{O}, \mathcal{P})$ is a (finite) disjoint union of neighborhoods of the form $\mathcal{N}^s({}_{\blacklozenge}f_i, \varepsilon)$ where ${}_{\blacklozenge}f_i = (f, \{\Theta_{f,x}\}_i)$.

For reference, we have the following diagram of maps:

$$
\begin{array}{ccc}
& \mathcal{N}^s({}_{\star}f, \varepsilon) & \\
{}^{q_{\blacklozenge}}\swarrow & & \searrow^{q} \\
\mathcal{N}^s({}_{\blacklozenge}f, \varepsilon) & & \mathcal{N}^s(f, \varepsilon) \\
\end{array}
$$

$$
\begin{array}{ccc}
q_{\dagger} & & \\
q_{\blacktriangledown} & \mathcal{N}^s([{}_{\blacklozenge}f], \varepsilon) & q_{\bullet} \\
& q_{\ddagger} \downarrow & \\
& \mathcal{N}^s({}_{\bullet}f, \varepsilon) &
\end{array}
$$

We now show that the action of identity maps is compatible with the C^s topology on $C_{\star\mathrm{Orb}}^r(\mathcal{O}, \mathcal{P})$. Let ${}_{\star}f = (f, \{\tilde{f}_x\}, \{\Theta_{f,x}\}) \in C_{\star\mathrm{Orb}}^r(\mathcal{O}, \mathcal{P})$ and let the orbifold $\mathcal{N}$ be an open neighborhood of the image of $f(\mathcal{O})$. Let $\varepsilon > 0$ be chosen so that if ${}_{\star}g = (g, \{\tilde{g}_x\}, \{\Theta_{g,x}\}) \in \mathcal{N}^s({}_{\star}f, \varepsilon)$, then $g(\mathcal{O}) \subset \mathcal{N}$. Let $I \in \mathcal{ID}_{\mathcal{N}}$ be an orbifold map lift of the identity over $\mathcal{N}$. Then by example 9, I has a representation as $I = (\mathrm{Id}, \{\tilde{w} \mapsto \gamma_z \cdot \tilde{w}\}, \{\delta \mapsto \gamma_z \delta \gamma_z^{-1}\})$.

Lemma 22. *With ${}_{\star}g \in \mathcal{N}^s({}_{\star}f, \varepsilon)$ as above, we have $I \circ {}_{\star}g \in \mathcal{N}^s(I \circ {}_{\star}f, \varepsilon)$. Thus, the local action of $\mathcal{ID}_{\mathcal{N}}$ on a neighborhood of ${}_{\star}f$ is continuous. Moreover, the map $I : \mathcal{N}^s({}_{\star}f, \varepsilon) \to \mathcal{N}^s(I \circ {}_{\star}f, \varepsilon)$, ${}_{\star}g \mapsto I \circ {}_{\star}g$, is a homeomorphism. In fact, the map I is a C^{∞} diffeomorphism.*

Proof. The homeomorphism claim is immediate from the definitions once one realizes that if $\tilde{U}_x$ is chosen as in definition 21, so that $\theta_{f(x)z} \circ \Theta_{f,x} = \theta_{g(x)z} \circ \Theta_{g,x}$, then for $\delta \in \Gamma_x$,

$$I \circ {}_{\star}f = (f, \{\gamma_z \cdot \tilde{f}_x\}, \{\gamma_z(\theta_{f(x)z} \circ \Theta_{f,x}(\delta))\gamma_z^{-1}\}) \text{ and,}$$

$$I \circ {}_{\star}g = (g, \{\gamma_z \cdot \tilde{g}_x\}, \{\gamma_z(\theta_{g(x)z} \circ \Theta_{g,x}(\delta))\gamma_z^{-1}\}).$$

The smoothness claims follow from lemma 51 in section 9. $\qquad\square$

Assuming theorem 1, we can now prove corollary 3.

Proof of corollary 3. Let ${}_{\blacklozenge}f = (f, \{\Theta_{f,x}\}) \in C_{\blacklozenge\mathrm{Orb}}^r(\mathcal{O}, \mathcal{P})$. For small enough $\varepsilon > 0$, it is clear that $\mathcal{Q}_{\star} = q_{\blacklozenge}^{-1}(\mathcal{N}^r({}_{\blacklozenge}f, \varepsilon)) = \bigsqcup \mathcal{N}^r({}_{\star}f_i, \varepsilon) \subset C_{\star\mathrm{Orb}}^r(\mathcal{O}, \mathcal{P})$, is a finite disjoint union of neighborhoods of the form $\mathcal{N}^r({}_{\star}f_i, \varepsilon)$ where ${}_{\star}f_i = (f, \{\tilde{f}_x\}_i, \{\Theta_{f,x}\})$. If $C_z = C_{\Gamma_z}(\theta_{f(x)z} \circ \Theta_{f,x}(\Gamma_x))$ (which $= C_{\Gamma_z}(\theta_{g(x)z} \circ \Theta_{g,x}(\Gamma_x))$), then lemmas 10 and 22 imply that $(\mathcal{ID}_{\mathcal{N}})_{\{C_z\}}$ acts smoothly and transitively on fibers of $\mathcal{Q}_{\star}$. Example 11 shows that this action is not necessarily free. To understand what happens under these circumstances, suppose that $I \in (\mathcal{ID}_{\mathcal{N}})_{\{C_z\}}$ fixes ${}_{\star}f = (f, \{\tilde{f}_x\}, \{\Theta_{f,x}\}) \in q_{\blacklozenge}^{-1}({}_{\blacklozenge}f)$. Let ${}_{\star}g = (g, \{\tilde{g}_x\}, \{\Theta_{g,x}\}) \in \mathcal{N}^r({}_{\star}f, \varepsilon)$. By lemma 22, $I \circ {}_{\star}g \in \mathcal{N}^r({}_{\star}f, \varepsilon)$. This shows that $q_{\blacklozenge}$ defines an orbifold chart. Since each $\mathcal{N}^r({}_{\star}f_i, \varepsilon)$ is an open manifold by theorem 1 and the action of $(\mathcal{ID}_{\mathcal{N}})_{\{C_z\}}$ is smooth, corollary 3 follows. $\quad\square$

Proposition 23. *The quotient map $q_\dagger : \mathcal{N}^s(_\bullet f, \varepsilon) \to \mathcal{N}^s([_\bullet f], \varepsilon)$, $_\bullet f = (f, \{\Theta_{f,x}\}) \mapsto [_\bullet f] = (f, \{[\Theta_{f,x}]\})$ is a local homeomorphism. In fact, it is the quotient map defined by the group action of $\mathcal{ID}_\mathcal{N}$ acting via $_\bullet f \mapsto I \circ {}_\bullet f$.*

Proof. It is clear from the definitions that $\mathcal{Q}_\dagger = q_\dagger^{-1}(\mathcal{N}^s([_\bullet f], \varepsilon))$ consists of finite disjoint union of neighborhoods of the form $\mathcal{N}^s(_\bullet f_i, \varepsilon)$ where $_\bullet f_i = (f, \{\eta_{x,i} \Theta_{f,x} \eta_{x,i}^{-1}\})$. The last statement follows by observing that $\mathcal{ID}_\mathcal{N}$ acts transitively on $\mathcal{Q}_\dagger$ and if $_\bullet f = I \circ {}_\bullet f$, for $I = (\mathrm{Id}, \{\tilde{y} \to \eta_x \cdot \tilde{y}\})$, then $I \in (\mathcal{ID}_\mathcal{N})_{\{C_z\}}$ where $C_z = C_{\Gamma_z}(\theta_{f(x)z} \circ \Theta_{f,x}(\Gamma_x))$. Since C_z also is $= C_{\Gamma_z}(\theta_{g(x)z} \circ \Theta_{g,x}(\Gamma_x))$ for any $_\bullet g \in \mathcal{N}^s(_\bullet f, \varepsilon)$, we see that any such I fixes pointwise the entire neighborhood $\mathcal{N}^s(_\bullet f, \varepsilon)$ and the result follows. $\square$

Later we will have need to refer to the following useful fact about the relation between $_\star f$ and maps $_\star g \in \mathcal{N}^s(_\star f, \varepsilon)$:

Lemma 24. *Let $_\star g = (g, \{\tilde{g}_x\}, \{\Theta_{g,x}\}) \in \mathcal{N}^s(_\star f, \varepsilon)$. Then for each $x \in \mathcal{O}$, $\Theta_{f,x} = \theta_{g(x)f(x)} \circ \Theta_{g,x} : \Gamma_x \to \Gamma_{f(x)}$. Moreover, $\tilde{f}_x(\tilde{x})$ and $\tilde{g}_x(\tilde{x})$ both belong to the same connected (closed) stratum $\tilde{V}_{f(x)}^{\Theta_{f,x}(\Gamma_x)} = \{\tilde{y} \in \tilde{V}_{f(x)} \mid \delta \cdot \tilde{y} = \tilde{y} \text{ for all } \delta \in \Theta_{f,x}(\Gamma_x)\}$.*

Proof. In definition 21 we may choose $z = f(x)$. This yields the stated equality of homomorphisms immediately. Recall that $\tilde{\psi}_{g(x)f(x)}$ denotes a lift of the inclusion map $\psi_{g(x)f(x)} : V_{g(x)} \hookrightarrow V_{f(x)}$ given in the definition of orbifold atlas. So, $\tilde{\psi}_{g(x)f(x)} \circ \tilde{g}_x : \tilde{U}_x \to \tilde{g}_x(\tilde{U}_x) \hookrightarrow \tilde{V}_{f(x)}$ is equivariant relative to $\theta_{g(x)f(x)} \circ \Theta_{g,x}$, which by hypothesis is the same as $\Theta_{f,x}$. Thus, for each $\gamma \in \Gamma_x$ we have

$$\tilde{\psi}_{g(x)f(x)} \circ \tilde{g}_x(\tilde{x}) = \tilde{\psi}_{g(x)f(x)} \circ \tilde{g}_x(\gamma \cdot \tilde{x}) = \left(\theta_{g(x)f(x)} \circ \Theta_{g,x}(\gamma)\right) \cdot \tilde{g}_x(\tilde{x})$$
$$= \Theta_{f,x}(\gamma) \cdot \tilde{g}_x(\tilde{x})$$

from which it follows that $\tilde{g}_x(\tilde{x}) \in \tilde{V}_{f(x)}^{\Theta_{f,x}(\Gamma_x)}$. $\square$

4. Applications to the Orbifold Diffeomorphism Group

In this section, we show how the discussion of the previous sections applies to orbifold diffeomorphisms. For simplicity, we will continue to assume that the orbifold $\mathcal{O}$ is compact. In [8], we studied the group of orbifold diffeomorphisms $\mathrm{Diff}_{\mathrm{Orb}}^r(\mathcal{O})$ and the reduced orbifold diffeomorphisms $\mathrm{Diff}_{\mathrm{red}}^r(\mathcal{O})$ showing that each carried the structure of a (topological) Banach/Fréchet manifold. In fact we expressed $\mathrm{Diff}_{\mathrm{red}}^r(\mathcal{O})$ as the quotient $\mathrm{Diff}_{\mathrm{Orb}}^r(\mathcal{O})/\mathcal{ID}$, where, of course, $\mathcal{ID} \subset \mathrm{Diff}_{\mathrm{Orb}}^r(\mathcal{O})$ represents the (finite) group of orbifold map lifts of the identity on $\mathcal{O}$.

For diffeomorphism groups, it is not hard to see that the group of complete orbifold diffeomorphisms $\mathrm{Diff}_{\star\mathrm{Orb}}^r(\mathcal{O})$ may be regarded as the same as $\mathrm{Diff}_{\mathrm{Orb}}^r(\mathcal{O})$ in much the same way that example 9 illustrated the correspondence $_\star\mathcal{ID} \leftrightarrow \mathcal{ID}$. This follows from the proof of corollary 1.2 in [8], where it is shown that if $f_1, f_2 \in \mathrm{Diff}_{\mathrm{Orb}}^r(\mathcal{O})$ represent the same reduced diffeomorphism $_\bullet f \in \mathrm{Diff}_{\mathrm{red}}^r(\mathcal{O})$, then $f_1 \circ f_2^{-1} \in \mathcal{ID}$. In the diffeomorphism case, one should note that since all homomorphisms $\Theta_{f,x}$ are actually isomorphisms and we assume isotropy groups act effectively, the behavior exhibited in lemmas 14 and 19 cannot occur. There can never be multiple $\Theta_{f,x}$'s corresponding to a particular local lift $\tilde{f}_x$. Collecting the results of example 9 and lemmas 10, 17 and 22, and exploiting the fact that, in the case of diffeomorphism groups, we have a *global* (C^∞-) smooth action of $\mathcal{ID}$, we get the following algebraic and topological structure result.

Theorem 25. *Let $\mathcal{O}$ be a compact smooth C^∞ orbifold. Then the following sequences are exact:*

$$1 \longrightarrow \mathcal{ID} \longrightarrow \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O}) \longrightarrow \mathrm{Diff}^r_{\mathrm{red}}(\mathcal{O}) \longrightarrow 1$$

$$1 \longrightarrow C(\mathcal{ID}) \longrightarrow \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O}) \longrightarrow \mathrm{Diff}^r_{\blacklozenge\mathrm{Orb}}(\mathcal{O}) \longrightarrow 1$$

$$1 \longrightarrow {}_\blacklozenge\mathcal{ID} = \mathcal{ID}/C(\mathcal{ID}) \longrightarrow \mathrm{Diff}^r_{\blacklozenge\mathrm{Orb}}(\mathcal{O}) \longrightarrow \mathrm{Diff}^r_{\mathrm{red}}(\mathcal{O}) \longrightarrow 1$$

where $C(\mathcal{ID})$ denotes the center of $\mathcal{ID}$. Moreover, each of the diffeomorphism groups $\mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$, $\mathrm{Diff}^r_{\blacklozenge\mathrm{Orb}}(\mathcal{O})$ and $\mathrm{Diff}^r_{\mathrm{red}}(\mathcal{O})$ carries the structure of a smooth C^∞ Banach $(r < \infty)$/Fréchet $(r = \infty)$ manifold.

Theorem 26. *Each of the diffeomorphism groups $\mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$, $\mathrm{Diff}^r_{\blacklozenge\mathrm{Orb}}(\mathcal{O})$ and $\mathrm{Diff}^r_{\mathrm{red}}(\mathcal{O})$ is a topological group. That is, composition and inversion are continuous. Furthermore, when $r = \infty$, $\mathrm{Diff}^\infty_{\mathrm{Orb}}(\mathcal{O})$, $\mathrm{Diff}^\infty_{\blacklozenge\mathrm{Orb}}(\mathcal{O})$ and $\mathrm{Diff}^\infty_{\mathrm{red}}(\mathcal{O})$ are convenient Fréchet Lie groups.*

Proof. For $0 < r < \infty$, the group multiplication $\mu(f, g) = f \circ g$ and inversion $\mathrm{inv}(f) = f^{-1}$ in the diffeomorphism group $\mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$, corresponds to composition and inversion of the C^r local equivariant lifts. These operations are known only to be C^0. This follows by the so-called Omega lemma [1, 24, 28], a suitable version of which is stated as lemma 53 for completeness. Thus, $\mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$ is a topological group. The structure result of theorem 25 then yields the topological group structure for $\mathrm{Diff}^r_{\blacklozenge\mathrm{Orb}}(\mathcal{O})$ and $\mathrm{Diff}^r_{\mathrm{red}}(\mathcal{O})$.

For $r = \infty$, by lemma 52, group multiplication $\mu(f, g) = f \circ g$ is smooth since $\mathrm{Diff}^\infty_{\mathrm{Orb}}(\mathcal{O})$ is an open submanifold of $C^\infty_{\mathrm{Orb}}(\mathcal{O}, \mathcal{O})$ by [8, section 7]. To show that inversion is smooth, we use the argument given in [22, Theorem 43.1]. Let $c = (c^t, \{\tilde{c}^t_x\}) : \mathbb{R} \to \mathrm{Diff}^\infty_{\mathrm{Orb}}(\mathcal{O}) \subset C^\infty_{\mathrm{Orb}}(\mathcal{O}, \mathcal{O})$ be a smooth curve. Then, in a local orbifold chart, by corollary 50, the mapping $\tilde{c}^\wedge_x : (0, 1) \times \tilde{U}_x \to \tilde{V}_z$ is smooth and $(\mathrm{inv} \circ \tilde{c}_x)^\wedge$ satisfies the finite dimensional implicit equation $\tilde{c}^\wedge_x(t, (\mathrm{inv} \circ \tilde{c}_x)^\wedge(t, \tilde{y})) = \tilde{y}$ for all $t \in \mathbb{R}$ and $\tilde{y} \in \tilde{U}_x$. By the finite dimensional implicit function theorem, $(\mathrm{inv} \circ \tilde{c}_x)^\wedge$ is smooth in $(t, \tilde{y})$. Hence, by corollary 50, inv maps smooth curves to smooth curves and is thus smooth. This shows that $\mathrm{Diff}^\infty_{\mathrm{Orb}}(\mathcal{O})$ is a convenient Fréchet Lie group and thus, by the structure results of theorem 25, so are $\mathrm{Diff}^\infty_{\blacklozenge\mathrm{Orb}}(\mathcal{O})$ and $\mathrm{Diff}^\infty_{\mathrm{red}}(\mathcal{O})$. $\square$

5. Why Non-Orbifold Structure Stratifications Arise

In this section, we wish to give an example on why non-orbifold structure stratifications arise in the topological structure of our orbifold maps. We first recall a definition of stratification in the infinite-dimensional setting. We will use the definition found in [14] or [9] for infinite-dimensional stratifications although we do not need the full generality presented in these references. In our case, each point with a stratified neighborhood has only a finite number of strata coming together.

Definition 27 ([9, 14]). Let X be a topological space and $\mathcal{A}$ a countable set with partial order $\prec$. A *partition* of X is a collection of non-empty pairwise disjoint subspaces $\{X_\alpha\}$ indexed by $\mathcal{A}$ such that $X = \cup_{\alpha \in \mathcal{A}} X_\alpha$. A partition $\{X_\alpha\}_{\alpha \in \mathcal{A}}$ is a *stratification* of X if

(1) each X_α is a submanifold when given the topology induced by X and,
(2) $X_\alpha \cap \overline{X_\beta} \neq \emptyset$, $\alpha \neq \beta$, then $\beta \prec \alpha$ and $X_\alpha \subset \overline{X_\beta}$.

The X_α are called the *strata* of the stratification and may have many connected components. Moreover, condition (2) implies that $\overline{X_\beta} - X_\beta \subset \cup_{\alpha \succ \beta} X_\alpha$.

Before we show how these stratifications arise, we first present a simple example to help motivate the discussion.

Example 28. Consider the situation described in Example 8: $\mathcal{O}$ is the orbifold $\mathbb{R}/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts on $\mathbb{R}$ via $x \to -x$ and $f : \mathcal{O} \to \mathcal{O}$ is the constant map $f \equiv 0$. The map $\tilde{f}_0 \equiv 0$ is a local equivariant lift of f at $x = 0$ using either of the homomorphisms $\Theta_{f,0} = \mathrm{Id}$ or $\Theta'_{f,0} \equiv e$. Of course, for $x \neq 0$, we set $\tilde{f}_x \equiv 0$ and $\Theta_{f,x} = \Theta'_{f,x} = $ the trivial homomorphism $\Gamma_x = e \mapsto e \in \Gamma_0 = \mathbb{Z}_2$. Thus, we have two complete orbifold maps $_\star f = (f, \{\tilde{f}_x\}, \{\Theta_{f,x}\})$ and $_\star f' = (f, \{\tilde{f}_x\}, \{\Theta'_{f,x}\})$ which cover the same orbifold map $f = (f, \{\tilde{f}_x\})$.

We need to first compute $\mathcal{N}^r(_\star f, \varepsilon)$. We will do this in detail since this is the first time we have done an explicit computation of this type. Using definition 21 and the notation there, let $_\star g \in \mathcal{N}^r(_\star f, \varepsilon)$. For all $x \in \mathcal{O}$ we may choose $z = 0$ and thus $\tilde{V}_z = \tilde{V}_0$ may be chosen to be the interval $(-\varepsilon, \varepsilon)$ as a chart about 0 in the target. There are two cases to consider: $x = 0$ and $x \neq 0$. For $x = 0$, let $\tilde{U}_0$ be any orbifold chart about 0. It follows that the local lift $\tilde{g}_0$ over $x = 0$ must take $0 \in \tilde{U}_0$ to $0 \in \tilde{V}_0$. To see this, suppose to the contrary that $\tilde{g}_0(0) = \tilde{y} \neq 0$. By definition 21, we must have the following equality of homomorphisms from $\mathbb{Z}_2 = \Gamma_0$ to Γ_0:

$$\theta_{f(0)0} \circ \Theta_{f,0} = \theta_{g(0)0} \circ \Theta_{g,0} \Longleftrightarrow$$
$$\mathrm{Id} = \theta_{y0} \circ \Theta_{g,0}$$

However, $\Theta_{g,0} : \Gamma_0 \to \Gamma_y = \{e\}$ has nontrivial kernel which contradicts the last line above. We thus may conclude that for $x = 0$, $\tilde{g}_0(0) = 0$ and $\Theta_{g,0} = \Theta_{f,0} = \mathrm{Id}$. From $\Theta_{g,0} = \mathrm{Id}$, it follows that the local lift $\tilde{g}_0$ must be an odd function. For $x \neq 0$, there is no restriction on $\tilde{g}_x$ arising from equivariance since $\Gamma_x = \{e\}$ and $\theta_{f(x)0} \circ \Theta_{f,x} = \theta_{g(x)0} \circ \Theta_{g,x} : \Gamma_x \to \Gamma_0$ will always be the trivial homomorphism $e \mapsto e$. Putting this all together we have shown that

$$q\left(\mathcal{N}^r(_\star f, \varepsilon)\right) = \{g \in C^r_{\mathrm{Orb}}(\mathcal{O}) \mid \|\tilde{g}_x\| < \varepsilon \text{ and } \tilde{g}_0 \text{ is an odd function}\}.$$

We now use a similar argument to compute $\mathcal{N}^r(_\star f', \varepsilon)$. Let $_\star g' \in \mathcal{N}^r(_\star f', \varepsilon)$. For $x \neq 0$, $\Theta'_{f,x} = \Theta_{f,x}$, so we conclude as above that there is no restriction on $\tilde{g}'_x$ arising from equivariance. On the other hand, for $x = 0$ we must have the equality of homomorphisms $\theta_{f(0)0} \circ \Theta'_{f,0} = \theta_{g'(0)0} \circ \Theta_{g',0} : \Gamma_0 \to \Gamma_0$. Since $\Theta'_{f,0} \equiv e$, injectivity of $\theta_{g'(0)0}$ implies that $\Theta_{g',0} \equiv e$. Thus, there is no restriction on $\tilde{g}'_0$ arising from equivariance either and we can conclude that

$$q\left(\mathcal{N}^r(_\star f', \varepsilon)\right) = \{g' \in C^r_{\mathrm{Orb}}(\mathcal{O}) \mid \|\tilde{g}'_x\| < \varepsilon\}.$$

Here, it is clear that $q\left(\mathcal{N}^r(_\star f, \varepsilon)\right)$ is a proper subset (later, a submanifold) of $q\left(\mathcal{N}^r(_\star f', \varepsilon)\right)$, and that any orbifold map $g \in \mathcal{N}^r(f, \varepsilon)$ must be in $q\left(\mathcal{N}^r(_\star f', \varepsilon)\right)$ so that the topological structure of a neighborhood of f is completely determined from only an understanding of the topological structure of $q\left(\mathcal{N}^r(_\star f', \varepsilon)\right)$ which in turn is determined by the structure of $\mathcal{N}^r(_\star f', \varepsilon)$, which will be shown to be a manifold.

Unfortunately, in general, the topological structure of a neighborhood of an orbifold map f is rarely determined completely by the topological structure of a

single neighborhood of one of its complete orbifold lifts $q^{-1}(f)$. This is illustrated in the next example.

Example 29. Let $\mathcal{O} = \mathbb{R}/\mathbb{Z}_2$ with $\mathbb{Z}_2$ acting with generator α, where $\alpha \cdot x = -x$ as above. Let $\mathcal{P} = \mathbb{R}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ where $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle j, k \mid j^2 = k^2 = 1 = [j, k] \rangle$ with the action defined by $j \cdot (x, y, z) = (-x, y, -z)$ and $k \cdot (x, y, z) = (-x, -y, z)$. Consider $_\bullet f \in C_{\mathrm{red}}^r(\mathcal{O}, \mathcal{P})$ defined by $_\bullet f(y_1) = (y_1, 0, 0)$ and choose the orbifold map $f \in q_\bullet^{-1}(_\bullet f)$ given by $f = (f, \{\tilde{y}_1 \mapsto (\tilde{y}_1, 0, 0)\})$. That is, for each $x \in \mathcal{O}$, the local lift $\tilde{f}_x(\tilde{y}_1) = (\tilde{y}_1, 0, 0)$ on $\tilde{U}_x$. Since Γ_x is trivial when $x \neq 0$ and $\Gamma_0 = \mathbb{Z}_2$, there are precisely two complete maps in $q^{-1}(f)$:

$$
\begin{aligned}
_\star f &= (f, \{\tilde{y}_1 \mapsto (\tilde{y}_1, 0, 0)\}, \{\Theta_{f,0} : \alpha \mapsto j\}) \\
_\star f' &= (f, \{\tilde{y}_1 \mapsto (\tilde{y}_1, 0, 0)\}, \{\Theta'_{f,0} : \alpha \mapsto k\}).
\end{aligned}
$$

Note that since $\Theta_{f,x}$ is the trivial homomorphism $e \mapsto e$ for all $x \neq 0$, we have only indicated the two possible homomorphisms at $x = 0$, namely, $\Theta_{f,0}, \Theta'_{f,0} : \Gamma_0 = \mathbb{Z}_2 \to \Gamma_{(0,0,0)} = \mathbb{Z}_2 \times \mathbb{Z}_2$.

We will proceed as in example 28 and first compute $\mathcal{N}^r(_\star f, \varepsilon)$. Let $_\star g \in \mathcal{N}^r(_\star f, \varepsilon)$. Then $_\star g$ has a representation

$$
_\star g = (g, \{\tilde{g}_x = (\tilde{y}_1 + (\tilde{g}_1)_x(\tilde{y}_1), (\tilde{g}_2)_x(\tilde{y}_1), (\tilde{g}_3)_x(\tilde{y}_1)\}, \{\Theta_{g,x}\}).
$$

For $x \neq 0$ we have $\Gamma_x = \{e\}$ so, like before, there is no restriction on $(\tilde{g}_i)_x, i = 1, 2, 3$ arising from equivariance. Thus, we focus on lifts $(\tilde{g}_i)_0$ over a chart $\tilde{U}_0$ about $x = 0$. We may assume that $\tilde{V}_z = \tilde{V}_{\vec{0}}$ where we have shortened the subscript $(0, 0, 0)$ to $\vec{0} \in \mathbb{R}^3$. We will continue to do this for the remainder of this example.

By lemma 24, $\tilde{g}_0(0) \in \tilde{V}_{\vec{0}}^{\{e,j\}} = y$-axis and $\Theta_{g,0} : \Gamma_0 \to \Gamma_{g(0)}$ is $\alpha \mapsto j$. We now compute

$$
\tilde{g}_0(\alpha \cdot \tilde{y}_1) = \tilde{g}_0(-y_1) = (-\tilde{y}_1 + (\tilde{g}_1)_0(-\tilde{y}_1), (\tilde{g}_2)_0(-\tilde{y}_1), (\tilde{g}_3)_0(-\tilde{y}_1)).
$$

On the other hand,

$$
\begin{aligned}
\tilde{g}_0(\alpha \cdot \tilde{y}_1) &= \Theta_{g,0}(\alpha) \cdot \tilde{g}_0(y_1) = j \cdot \tilde{g}_0(y_1) \\
&= (-\tilde{y}_1 - (\tilde{g}_1)_0(\tilde{y}_1), (\tilde{g}_2)_0(\tilde{y}_1), -(\tilde{g}_3)_0(\tilde{y}_1)).
\end{aligned}
$$

Thus,

$$
q(\mathcal{N}^r(_\star f, \varepsilon)) = \{g \in C_{\mathrm{Orb}}^r(\mathcal{O}, \mathcal{P}) \mid
$$
$$
\|\tilde{g}_x - \tilde{f}_x\| < \varepsilon \text{ with } (\tilde{g}_1)_0, (\tilde{g}_3)_0 \text{ odd functions and } (\tilde{g}_2)_0 \text{ an even function}\}.
$$

Similarly, we have

$$
q(\mathcal{N}^r(_\star f', \varepsilon)) = \{g' \in C_{\mathrm{Orb}}^r(\mathcal{O}, \mathcal{P}) \mid
$$
$$
\|\tilde{g}'_x - \tilde{f}_x\| < \varepsilon \text{ with } (\tilde{g}'_1)_0, (\tilde{g}'_2)_0 \text{ odd functions and } (\tilde{g}'_3)_0 \text{ an even function}\}.
$$

Thus, the corresponding neighborhood of the orbifold map f is the union of two sets $\mathcal{N}^r(f, \varepsilon) = q(\mathcal{N}^r(_\star f, \varepsilon)) \cup q(\mathcal{N}^r(_\star f', \varepsilon))$ each of which will later be shown to carry a Banach/Fréchet manifold structure. Their intersection is along the submanifold

$$
\mathcal{H} = q(\mathcal{N}^r(_\star f, \varepsilon)) \cap q(\mathcal{N}^r(_\star f', \varepsilon)) =
$$
$$
\{h \in C_{\mathrm{Orb}}^r(\mathcal{O}, \mathcal{P}) \mid \|\tilde{h}_x - \tilde{f}_x\| < \varepsilon \text{ with } \tilde{h}_0(\tilde{y}_1) = (\tilde{y}_1 + (\tilde{h}_1)_0(\tilde{y}_1), 0, 0)
$$
$$
\text{where } (\tilde{h}_1)_0 \text{ is an odd function}\}.
$$

Thus, the neighborhood $\mathcal{N}^r(f, \varepsilon)$ has a stratified structure (see figure 1): Just let $\mathcal{A} = \{\alpha, \beta, \gamma\}$ with partial order $\beta \prec \alpha$, $\gamma \prec \alpha$ and define $X = \mathcal{N}^r(f, \varepsilon)$, $X_\alpha = \mathcal{H}$, $X_\beta = q(\mathcal{N}^r(\star f, \varepsilon)) - \mathcal{H}$, and $X_\gamma = q(\mathcal{N}^r(\star f', \varepsilon)) - \mathcal{H}$. Moreover, since $\mathcal{N}^r(f, \varepsilon) - \mathcal{H}$ is not connected we see that this stratified structure is not that of an orbifold structure as removal of the singular set of an orbifold never disconnects a connected component of the orbifold [5, 6]. Furthermore, if we let $\mathcal{N}$ denote an open neighborhood of the image $f(\mathcal{O})$, then from [8] a neighborhood of the reduced orbifold map $\bullet f$ is given by $\mathcal{N}^r(\bullet f, \varepsilon) = \mathcal{N}^r(f, \varepsilon)/\mathfrak{ID}_\mathcal{N}$ where $\mathfrak{ID}_\mathcal{N}$ acts in such a way that the quotient map restricts on each stratum to give a smooth orbifold chart (see proof of corollary 5 which appears at the end of section 8). Thus, $\mathcal{N}^r(\bullet f, \varepsilon)$ has a non-orbifold structure stratification also.
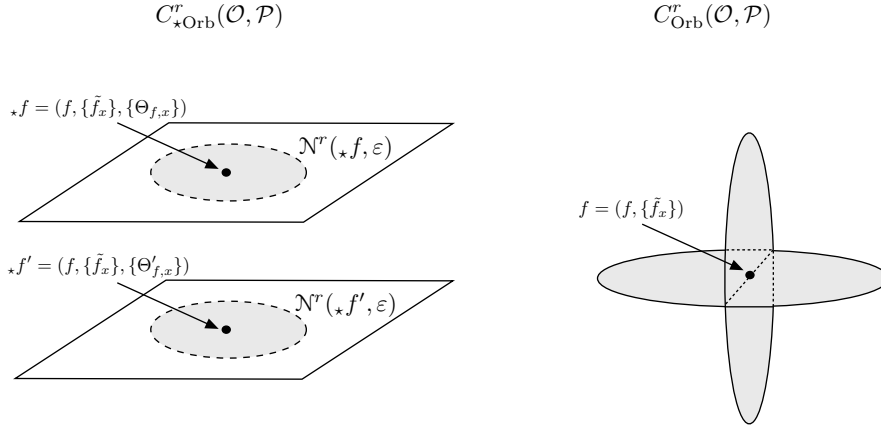
$$C^r_{\star\mathrm{Orb}}(\mathcal{O}, \mathcal{P}) \qquad\qquad C^r_{\mathrm{Orb}}(\mathcal{O}, \mathcal{P})$$

FIGURE 1. A stratified neighborhood

6. The Tangent Orbibundle, Pullbacks and Orbisections

The tangent orbibundle. We recall the definition of the tangent orbibundle of a smooth C^∞ orbifold.

Definition 30. Let $\mathcal{O}$ be an n–dimensional C^∞ orbifold. The *tangent orbibundle* of $\mathcal{O}$, $p : T\mathcal{O} \to \mathcal{O}$, is the C^∞ orbibundle defined as follows. If $(\tilde{U}_x, \Gamma_x)$ is an orbifold chart around $x \in \mathcal{O}$ then $p^{-1}(U_x) \cong (\tilde{U}_x \times \mathbb{R}^n)/\Gamma_x$ where Γ_x acts on $\tilde{U}_x \times \mathbb{R}^n$ via $\gamma \cdot (\tilde{y}, \tilde{v}) = (\gamma \cdot \tilde{y}, d\gamma_{\tilde{y}}(\tilde{v}))$. In keeping with tradition, we denote the fiber $p^{-1}(x)$ over $x \in U_x$ by $T_x\mathcal{O} \cong \mathbb{R}^n/\Gamma_x$. Note that, in general, if Γ_x is non-trivial then $T_x\mathcal{O}$ will be a convex cone rather than a vector space. Locally we have the diagram:

$$
\begin{array}{ccc}
T\tilde{U}_x \cong \tilde{U}_x \times \mathbb{R}^n & \xrightarrow{\ \Pi_x\ } & (\tilde{U}_x \times \mathbb{R}^n)/\Gamma_x \\
{\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle p} \\
\tilde{U}_x & \xrightarrow{\ \pi_x\ } & U_x
\end{array}
$$

where $\mathrm{pr}_1 : \tilde{U}_x \times \mathbb{R}^n \to \tilde{U}_x$ denotes the projection onto the first factor $(\tilde{y}, \tilde{v}) \mapsto \tilde{y}$ (which is a specific choice of lift of p).

Pulling back an orbibundle. The definition of the pullback of an orbibundle depends crucially on the notion of orbifold map. In simple examples, we will see that a unique notion of pullback exists only when using complete orbifold maps. On the other hand, we will see that once one has a pullback bundle defined via a complete orbifold map $_\star f$, there is no difference between the notion of an orbisection and a complete orbisection. Not surprisingly, if one tries to define a useful notion of reduced or complete reduced orbisection one loses the vector space structure on the space of such sections. As in the case of the tangent orbibundle, the pullback bundle will be an example of the more general notion of a linear orbibundle given in [7].

Definition 31. Let $\mathcal{O}, \mathcal{P}$ be C^∞ orbifolds of dimension n and m, respectively. Given $_\star f \in C^r_{\star\mathrm{Orb}}(\mathcal{O}, \mathcal{P})$ we define the pullback of the tangent orbibundle to $\mathcal{P}$ by $_\star f$, $_\star f^*(T\mathcal{P})$ as follows: Let $_\star f = (f, \{\tilde{f}_x\}, \{\Theta_{f,x}\})$ and let $\tilde{U}_x$ and $\tilde{V}_{f(x)}$ be orbifold charts about $x \in \mathcal{O}$ and $f(x) \in \mathcal{P}$ respectively. Define the pullback $_\star f^*(T\mathcal{P})$ to be the orbifold with charts of the form (a fibered product)

$$\tilde{U}_x \times_{\tilde{V}_{f(x)}} T\tilde{V}_{f(x)} = \{(\tilde{y}, \tilde{\xi}) \in \tilde{U}_x \times T\tilde{V}_{f(x)} \mid \tilde{f}_x(\tilde{y}) = \mathrm{pr}_1(\tilde{\xi})\}$$

where $\tilde{f}_x : \tilde{U}_x \to \tilde{V}_{f(x)}$ and $\mathrm{pr}_1 : T\tilde{V}_{f(x)} \to \tilde{V}_{f(x)}$ is the tangent bundle projection. If we write $\tilde{\xi} = [\tilde{f}_x(\tilde{y}), \tilde{v}] \in T\tilde{V}_{f(x)} = \tilde{V}_{f(x)} \times \mathbb{R}^m$, the action of Γ_x is specified in local coordinates by:

$$\gamma \cdot (\tilde{y}, \tilde{\xi}) = (\gamma \cdot \tilde{y}, \Theta_{f,x}(\gamma) \cdot \tilde{\xi}) = (\gamma \cdot \tilde{y}, [\Theta_{f,x}(\gamma) \cdot \tilde{f}_x(\tilde{y}), d(\Theta_{f,x}(\gamma))_{\tilde{f}_x(\tilde{y})} \cdot \tilde{v}])$$
$$= (\gamma \cdot \tilde{y}, [\tilde{f}_x(\gamma \cdot \tilde{y}), d(\Theta_{f,x}(\gamma))_{\tilde{f}_x(\tilde{y})} \cdot \tilde{v}]) \in \tilde{U}_x \times_{\tilde{V}_{f(x)}} T\tilde{V}_{f(x)}$$

where $\tilde{y} \in \tilde{U}_x$ and $\tilde{v} \in \mathrm{pr}_1^{-1}(\tilde{f}_x(\tilde{y}))$. Also, we let $\mathrm{pr}_2 : T\tilde{V}_{f(x)} \cong \tilde{V}_{f(x)} \times \mathbb{R}^m \to \mathbb{R}^m$, $\tilde{\xi} \mapsto \tilde{v}$ be the fiber projection. This gives $_\star f^*(T\mathcal{P})$ the structure of a smooth C^r m-dimensional linear orbibundle over $\mathcal{O}$. In an abuse of notation, $p : {_\star f^*(T\mathcal{P})} \to \mathcal{O}$ will denote the orbibundle projection. Denote the fiber over x, by $p^{-1}(x) = {_\star f^*(T\mathcal{P})_x}$. In local coordinates, we have the diagram (all vertical arrows are quotient maps by respective group actions):

$$\begin{array}{c}
\tilde{U}_x \times_{\tilde{V}_{f(x)}} T\tilde{V}_{f(x)} \xrightarrow{\quad \tilde{F}_x \quad} T\tilde{V}_{f(x)} \cong (\tilde{V}_{f(x)} \times \mathbb{R}^m)
\end{array}$$

The map $\tilde{F}_x : \tilde{U}_x \times_{\tilde{V}_{f(x)}} T\tilde{V}_{f(x)} \to T\tilde{V}_{f(x)}$ given by $\tilde{F}_x(\tilde{y},\tilde{\xi}) = \tilde{\xi}$ induces a map $F :$ $_\star f^*(TV_{f(x)}) \to T\tilde{V}_{f(x)}/\Theta_{f,x}(\Gamma_x)$ defined by $F(y,\xi) = \Pi_{\Theta_{f,x}} \circ \tilde{F}_x \circ \Pi_x^{-1}(y,\xi)$. This is well defined since, for any $\gamma \in \Gamma_x$, $\tilde{F}_x(\gamma \cdot (\tilde{y},\tilde{\xi})) = \tilde{F}_x(\gamma \cdot \tilde{y}, \Theta_{f,x}(\gamma) \cdot \tilde{\xi}) = \Theta_{f,x}(\gamma) \cdot \tilde{\xi}$ and $\Pi_{\Theta_{f,x}}(\Theta_{f,x}(\gamma) \cdot \tilde{\xi}) = \Pi_{\Theta_{f,x}}(\tilde{\xi})$.

Note that the pullback is defined only if we have all the information contained in both the choices of local lifts $\{\tilde{f}_x\}$ and the choices of the homomorphisms $\Theta_{f,x} \in \mathrm{Hom}(\Gamma_x, \Gamma_{f(x)})$. That is, all of the information of a complete orbifold map is used. As an illustration of the necessity for needing to use complete orbifold maps to define pullbacks we give two examples. The first example shows that, unless complete orbifold maps are used, the pullback orbibundle is not well-defined even up to a reasonable notion of equivalence.

Example 32. Consider the situation from example 28: $\mathcal{O} = \mathcal{P}$ is the orbifold $\mathbb{R}/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts on $\mathbb{R}$ via $x \to -x$ and $f : \mathcal{O} \to \mathcal{P}$ is the constant map $f \equiv 0$. Note that $T\mathcal{O} = T\mathcal{P} = \mathbb{R}^2/\mathbb{Z}_2$ where the generator α of $\mathbb{Z}_2$ acts via $(x,y) \mapsto (-x,-y)$ and the bundle projection p is just projection onto the first factor. We note that for $x \neq 0$, $p^{-1}(x) = \mathbb{R}$ and that $p^{-1}(0) = \mathbb{R}/\mathbb{Z}_2$. Let $_\star f$ and $_\star f'$ be the two complete orbifold maps from example 28 which cover the orbifold map $f = (f, \{\tilde{f}_x\})$. Then we claim that

$$_\star f^*(T\mathcal{P}) \cong T\mathcal{O}$$

while

$$_\star f'^*(T\mathcal{P}) \cong \mathcal{O} \times \mathbb{R}$$

To see this we work in local coordinates: Since $\tilde{f}_x \equiv 0$, we may take $\tilde{V}_0$ as a chart about $f(x)$ for all x. Thus for each x,

$$\tilde{U}_x \times_{\tilde{V}_0} T\tilde{V}_0 = \{(\tilde{y}, \tilde{\xi}) \in \tilde{U}_x \times T\tilde{V}_0 \mid 0 = \mathrm{pr}_1(\tilde{\xi})\}$$
$$= \tilde{U}_x \times T_0\tilde{V}_0 \cong \tilde{U}_x \times \mathbb{R}$$

Now for $x \neq 0$, $\Gamma_x = \{e\}$ and so the action of Γ_x on $\tilde{U}_x \times_{\tilde{V}_0} T\tilde{V}_0$ is necessarily trivial. If we denote the orbibundle projections $p : {}_\star f^*(T\mathcal{P}) \to \mathcal{O}$ and $p' : {}_\star f'^*(T\mathcal{P}) \to \mathcal{O}$, then $p^{-1}(U_x) = p'^{-1}(U_x) \cong \tilde{U}_x \times \mathbb{R}$. On the other hand, for $x = 0$, since $\Theta_{f,0}(\alpha) = \alpha$ and $\Theta'_{f,0}(\alpha) = e$ we see that

$p^{-1}(U_0) \cong (\tilde{U}_0 \times \mathbb{R})/\Gamma_0$ where the action of Γ_0 is
$$\alpha \cdot (\tilde{y}, \tilde{v}_0) = (\alpha \cdot \tilde{y}, d(\Theta_{f,0}(\alpha))_{\tilde{f}_0(\tilde{y})} \cdot \tilde{v}_0) = (\alpha \cdot \tilde{y}, d(-\mathrm{Id})_0 \cdot \tilde{v}_0) = (-\tilde{y}, -\tilde{v}_0)$$

while

$p'^{-1}(U_0) \cong (\tilde{U}_0 \times \mathbb{R})/\Gamma_0$ where the action of Γ_0 is
$$\alpha \cdot (\tilde{y}, \tilde{v}_0) = (\alpha \cdot \tilde{y}, d(\Theta'_{f,0}(\alpha))_{\tilde{f}_0(\tilde{y})} \cdot \tilde{v}_0) = (\alpha \cdot \tilde{y}, d(\mathrm{Id})_0 \cdot \tilde{v}_0) = (-\tilde{y}, \tilde{v}_0)$$

which is enough to substantiate our claim. Note that these orbibundles are not equivalent in any reasonable sense.

To further illustrate the complexity involved in pulling back the tangent bundle by an orbifold map, the following is instructive.

Example 33. Consider the situation from example 29: $\mathcal{O} = \mathbb{R}/\mathbb{Z}_2$ with $\mathbb{Z}_2$ acting with generator α, where $\alpha \cdot x = -x$ as above and $\mathcal{P} = \mathbb{R}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ where $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle j, k \mid j^2 = k^2 = 1 = [j, k] \rangle$ with the action defined by $j \cdot (x, y, z) = (-x, y, -z)$ and $k \cdot (x, y, z) = (-x, -y, z)$. We consider the two complete orbifold maps ${}_\star f$ and ${}_\star f'$ from example 29 which cover the orbifold map $f = (f, \{\tilde{y}_1 \mapsto (\tilde{y}_1, 0, 0)\})$ where ${}_\bullet f(y_1) = (y_1, 0, 0)$. We have for all x, $\tilde{U}_x \times_{\tilde{V}_{f(x)}} T\tilde{V}_{f(x)} \cong \tilde{U}_x \times \mathbb{R}^3$. Like in example 32, for $x \neq 0$, $\Gamma_x = \{e\}$, and so the action of Γ_x on $\tilde{U}_x \times_{\tilde{V}_{f(x)}} T\tilde{V}_{f(x)}$ is necessarily trivial. If we denote, as before, the orbibundle projections $p : {}_\star f^*(T\mathcal{P}) \to \mathcal{O}$ and $p' : {}_\star f'^*(T\mathcal{P}) \to \mathcal{O}$, then $p^{-1}(U_x) = p'^{-1}(U_x) \cong \tilde{U}_x \times \mathbb{R}^3$. On the other hand, for $x = 0$, since $\Theta_{f,0}(\alpha) = j$ and $\Theta'_{f,0}(\alpha) = k$ we see that

$p^{-1}(U_0) \cong (\tilde{U}_0 \times \mathbb{R}^3)/\Gamma_0$ where the action of Γ_0 is
$$\alpha \cdot (\tilde{y}, \tilde{v}_0) = (\alpha \cdot \tilde{y}, dj_{\tilde{f}_0(\tilde{y})} \cdot \tilde{v}_0) = (-\tilde{y}, -(\tilde{v}_1)_0, (\tilde{v}_2)_0, -(\tilde{v}_3)_0)$$

where $\tilde{v}_0 = ((\tilde{v}_1)_0, (\tilde{v}_2)_0, (\tilde{v}_3)_0) \in \mathbb{R}^3$. Similarly,

$p'^{-1}(U_0) \cong (\tilde{U}_0 \times \mathbb{R}^3)/\Gamma_0$ where the action of Γ_0 is
$$\alpha \cdot (\tilde{y}, \tilde{v}_0) = (\alpha \cdot \tilde{y}, dk_{\tilde{f}_0(\tilde{y})} \cdot \tilde{v}_0) = (-\tilde{y}, -(\tilde{v}_1)_0, -(\tilde{v}_2)_0, (\tilde{v}_3)_0).$$

Although the pullback orbibundles ${}_\star f^*(T\mathcal{P})$ and ${}_\star f'^*(T\mathcal{P})$ are naturally isomorphic, we will later see that neighborhoods of the zero section are taken by the Riemannian exponential map to the neighborhoods $\mathcal{N}^r({}_\star f, \varepsilon)$ and $\mathcal{N}^r({}_\star f', \varepsilon)$ of example 29, respectively. This illustrates why it is necessary to use *complete* orbifold maps in order to fully understand the topological structure of a neighborhood $\mathcal{N}^r(f, \varepsilon)$ of an orbifold map.

Orbisections. We now define a natural notion of section of a linear orbibundle. For a definition, see for example [7].

Definition 34. A C^r *orbisection* of a m-dimensional linear orbibundle $p : \mathcal{E} \to \mathcal{O}$ is a C^r orbifold map $\sigma : \mathcal{O} \to \mathcal{E}$ such that $p \circ \sigma = \mathrm{Id}_{\mathcal{O}}$ and for any $x \in \mathcal{O}$ and chart U_x about x, we have $\mathrm{pr}_1 \circ \tilde{\sigma}_x = \mathrm{Id}_{\tilde{U}_x}$. That is, we take the *identity* lift of the identity $\mathrm{Id}_{\mathcal{O}}$ in $\tilde{U}_x$. Locally we have the diagram:

Note that the action of Γ_x on $\tilde{U}_x \times \mathbb{R}^m$ is given as part of the data defining $\mathcal{E}$. Although, in general, the class of complete orbifold maps is different from the class of orbifold maps, as in the case for diffeomorphisms (section 4), in the case of orbisections of the pullback of a tangent orbibundle, these notions coincide.

Proposition 35. *Let $_\star f : \mathcal{O} \to \mathcal{P}$ be a C^r complete orbifold map between C^∞ orbifolds and let $_\star f^*(T\mathcal{P})$ denote the pullback of the tangent orbibundle. Let $\sigma = (\sigma, \{\tilde{\sigma}_x\})$ be a C^r orbisection of $_\star f^*(T\mathcal{P})$. Then there is a unique homomorphism $\Theta_{\sigma,x}$ for which $\tilde{\sigma}_x$ is Γ_x equivariant. In other words, the set of orbisections can be identified with the set of complete orbisections $\sigma \leftrightarrow {}_\star\sigma$.*

Proof. Given an orbifold chart $\tilde{U}_x$ around x and an orbibundle chart for $_\star f^*(T\mathcal{P})$ with local product coordinates $(\tilde{y}, \tilde{\xi}) = (\tilde{y}, [\tilde{f}_x(\tilde{y}), \tilde{v}]) \in \tilde{U}_x \times_{\tilde{V}_{f(x)}} T\tilde{V}_{f(x)}$, the local action of Γ_x in these coordinates is given by $\gamma \cdot (\tilde{y}, \tilde{\xi}) = (\gamma \cdot \tilde{y}, \Theta_{f,x}(\gamma) \cdot \tilde{\xi}) = (\gamma \cdot \tilde{y}, [\Theta_{f,x}(\gamma) \cdot \tilde{f}_x(\tilde{y}), d(\Theta_{f,x}(\gamma))_{\tilde{f}_x(\tilde{y})} \cdot \tilde{v}])$. With respect to these local coordinates, $\tilde{\sigma}_x$ has the form $\tilde{\sigma}_x(\tilde{y}) = (\tilde{y}, [\tilde{f}_x(\tilde{y}), \tilde{s}_x(\tilde{y})])$ and if $\Theta_{\sigma,x} : \Gamma_x \to \Gamma_{\sigma(x)} = \Gamma_x$ is some homomorphism for which $\tilde{\sigma}_x$ is equivariant with respect to, then

$$\tilde{\sigma}_x(\gamma \cdot \tilde{y}) = (\gamma \cdot \tilde{y}, [\tilde{f}_x(\gamma \cdot \tilde{y}), \tilde{s}_x(\gamma \cdot \tilde{y})])$$
$$= \Theta_{\sigma,x}(\gamma) \cdot (\tilde{y}, [\tilde{f}_x(\tilde{y}), \tilde{s}_x(\tilde{y})])$$
$$= (\Theta_{\sigma,x}(\gamma) \cdot \tilde{y}, [(\Theta_{f,x}(\Theta_{\sigma,x}(\gamma)) \cdot \tilde{f}_x(\tilde{y}), d(\Theta_{f,x}(\Theta_{\sigma,x}(\gamma)))_{\tilde{f}_x(\tilde{y})} \cdot \tilde{s}_x(\tilde{y})])$$

Therefore, since Γ_x acts effectively on $\tilde{U}_x$, $\Theta_{\sigma,x}(\gamma) = \gamma$ and $\Theta_{\sigma,x} = \mathrm{Id} : \Gamma_x \to \Gamma_x$ for all $\gamma \in \Gamma_x$ and $x \in \mathcal{O}$. Furthermore, we get the equivariance relation $\tilde{s}_x(\gamma \cdot \tilde{y}) = d(\Theta_{f,x}(\gamma))_{\tilde{f}_x(\tilde{y})}(\tilde{s}_x(\tilde{y}))$. $\square$

Just as in the case of orbisections of the tangent orbibundle, the set of orbisections of the pullback tangent orbibundle carry a vector space structure.

Proposition 36. *Let* $_\star f \in C^r_{\star\mathrm{Orb}}(\mathcal{O}, \mathcal{P})$. *The set* $\mathcal{D}^r_{\mathrm{Orb}}(_\star f^*(T\mathcal{P}))$ *of* C^r *orbisections of the the pullback tangent orbibundle* $_\star f^*(T\mathcal{P})$ *is naturally a real vector space with the vector space operations being defined pointwise.*

Proof. The argument here is basically the same as the corresponding argument for orbisections of the tangent orbibundle [8]. Let $\sigma \in \mathcal{D}^r_{\mathrm{Orb}}(_\star f^*(T\mathcal{P}))$. Let $\tilde{\sigma}_x$ be the lift of σ. Then $\tilde{\sigma}_x(\tilde{y}) = (\tilde{y}, [\tilde{f}_x(\tilde{y}), \tilde{s}_x(\tilde{y})])$. By proposition 35, $\tilde{s}_x : \tilde{U}_x \to \mathbb{R}^m$ is such that $\tilde{s}_x(\gamma \cdot \tilde{y}) = d(\Theta_{f,x}(\gamma))_{\tilde{f}_x(\tilde{y})}(\tilde{s}_x(\tilde{y}))$. In particular, since $\tilde{x}$ is a fixed point of the Γ_x action on $\tilde{U}_x$, we have $\tilde{s}_x(\tilde{x}) = \tilde{s}_x(\gamma \cdot \tilde{x}) = d(\Theta_{f,x}(\gamma))_{\tilde{f}_x(\tilde{x})}(\tilde{s}_x(\tilde{x}))$. Thus $\tilde{s}_x(\tilde{x})$ is a fixed point of the (linear) action of Γ_x on $\mathbb{R}^m$ as defined in the pullback orbibundle. Note that the set of such fixed points forms a *vector subspace* of $\mathbb{R}^m$. As a result we may define a real vector space structure on $\mathcal{D}^r_{\mathrm{Orb}}(_\star f^*(T\mathcal{P}))$ as follows: For $\sigma_i \in \mathcal{D}^r_{\mathrm{Orb}}(_\star f^*(T\mathcal{P}))$, let $\tilde{\sigma}_{i,x}$ be local lifts at x as above. Define

$$(\sigma_1 + \sigma_2)(y) = \Pi_x\big((\tilde{\sigma}_{1,x} + \tilde{\sigma}_{2,x})(\tilde{y})\big) = \Pi_x\big((\tilde{y}, \tilde{s}_1(\tilde{y}) + \tilde{s}_2(\tilde{y}))\big) = \sigma_1(y) + \sigma_2(y)$$

$$(\lambda\sigma)(y) = \Pi_x\big((\lambda\tilde{\sigma}_x)(\tilde{y})\big) = \Pi_x\big((\tilde{y}, \lambda\tilde{s}(\tilde{y}))\big) = \lambda(\sigma(y))$$

$\square$

Proposition 37. *Let* $_\star f \in C^r_{\star\mathrm{Orb}}(\mathcal{O}, \mathcal{P})$ *with* $\mathcal{O}$ *compact (without boundary). The inclusion map* $\mathcal{D}^r_{\mathrm{Orb}}(_\star f^*(T\mathcal{P})) \hookrightarrow C^r_{\mathrm{Orb}}(\mathcal{O}, _\star f^*(T\mathcal{P}))$ *induces a separable Banach space structure on* $\mathcal{D}^r_{\mathrm{Orb}}(_\star f^*(T\mathcal{P}))$ *for* $1 \leq r < \infty$ *and a separable Fréchet space structure if* $r = \infty$.

Proof. The argument here is also similar to the corresponding argument for orbisections of the tangent orbibundle [8]. Let $\mathcal{D} = \{D_i\}_{i=1}^N$ be a cover of $f(\mathcal{O})$ by a finite number of compact orbifold charts over each of which the tangent orbibundle $T\mathcal{P}$ is trivialized. Then the collection $\mathcal{C} = \{C_i = f^{-1}(D_i)\}$ is a finite cover of $\mathcal{O}$ by compact subsets. By reindexing and shrinking D_i if necessary, we may assume each C_i is connected and is contained in a orbifold chart of $\mathcal{O}$ and so, $C_i \cong \tilde{C}_i/\Gamma_i$. Let $\tilde{C}_i \times_{\tilde{D}_i} T\tilde{D}_i \cong \tilde{C}_i \times \mathbb{R}^m$ be the corresponding orbifold charts for $_\star f^*(T\mathcal{P})$ with action of $\Gamma_i : \gamma \cdot (\tilde{y}, \tilde{\xi}) = (\gamma \cdot \tilde{y}, [\Theta_{f,i}(\gamma) \cdot \tilde{f}_x(\tilde{y}), d(\Theta_{f,i}(\gamma))_{\tilde{f}_x(\tilde{y})} \cdot \tilde{v}])$. Let $V_{i,r} = C^r(\tilde{C}_i, \mathbb{R}^m)$ for $i = 1, \ldots, N$ and $0 \leq r \leq \infty$ with topology of uniform convergence of derivatives of order $\leq r$. This is a Banach space for finite r and a Fréchet space for $r = \infty$. For finite r, let $\| \cdot \|_{i,r}$ be a C^r norm on $V_{i,r}$. Define a linear map $L : \mathcal{D}^r_{\mathrm{Orb}}(_\star f^*(T\mathcal{P})) \to \bigoplus_{i=1}^N V_{i,r}$ by

$$L(\sigma) = (\mathrm{pr}_2(\tilde{\chi}_1\tilde{\sigma}), \ldots, \mathrm{pr}_2(\tilde{\chi}_N\tilde{\sigma}))$$

where $\chi_i \in C^r_{\mathrm{Orb}}(\mathcal{O}, [0,1])$, $i = 1, \ldots, N$, is a partition of unity subordinate to the cover $\mathcal{C}$ ([8, proposition 6.1]) and $\mathrm{pr}_2 : \tilde{C}_i \times \mathbb{R}^m \to \mathbb{R}^m$ is bundle projection onto the second factor. Continuity of L is immediate from the definitions of the C^r topology on $\mathcal{D}^r_{\mathrm{Orb}}(_\star f^*(T\mathcal{P}))$ and the topology on $\bigoplus_{i=1}^N V_{i,r}$. Moreover, given a neighborhood of the zero section $\mathbf{0} \in \mathcal{D}^r_{\mathrm{Orb}}(_\star f^*(T\mathcal{P}))$ of the form $\mathcal{N}^r(\mathbf{0}, \varepsilon_i; \mathcal{C})$, it is apparent that there is a neighborhood of the zero section $\mathbf{0}$ in $\bigoplus_{i=1}^N V_{i,r}$ of the form $\max\{\|s_1\|_{1,r}, \ldots, \|s_N\|_{N,r}\} < \delta$ where $\delta \leq \min\{\varepsilon_1, \ldots, \varepsilon_N\}$ contained in $L(\mathcal{N}^r(\mathbf{0}, \varepsilon_i; \mathcal{C}))$. Thus, with the subspace topology on $L(\mathcal{D}^r_{\mathrm{Orb}}(_\star f^*(T\mathcal{P})))$, $L : \mathcal{D}^r_{\mathrm{Orb}}(_\star f^*(T\mathcal{P})) \to L(\mathcal{D}^r_{\mathrm{Orb}}(_\star f^*(T\mathcal{P})))$ is a linear homeomorphism. Since $\mathcal{D}^r_{\mathrm{Orb}}(_\star f^*(T\mathcal{P})) \subset C^r_{\mathrm{Orb}}(\mathcal{O}, _\star f^*(T\mathcal{P}))$ is a closed subset, we see that $L(\mathcal{D}^r_{\mathrm{Orb}}(_\star f^*(T\mathcal{P})))$ is a closed subspace of the direct sum and thus

$\mathcal{D}^r_{\mathrm{Orb}}(\star f^*(T\mathcal{P}))$ inherits a Banach space structure if $r < \infty$ and a Fréchet space structure if $r = \infty$.

$\square$

The following is the analogue of the notion of admissible tangent vector as defined in [8].

Definition 38. Let $\mathcal{O}, \mathcal{P}$ and f be as in proposition 37. Let $x \in \mathcal{O}$. Denote by $A_x(\star f^*(T\mathcal{P}))$ the set of *admissible vectors at x*

$$A_x(\star f^*(T\mathcal{P})) = \left\{ v \in \star f^*(T\mathcal{P})_x \mid (x, v) = \sigma(x) \text{ for some } \sigma \in \mathcal{D}^0_{\mathrm{Orb}}(\star f^*(T\mathcal{P})) \right\}$$

By proposition 36, $A_x(\star f^*(T\mathcal{P}))$ is a vector space for each x, and a suborbifold of $\star f^*(T\mathcal{P})_x$. The *admissible pullback bundle of $T\mathcal{P}$* is the subset $A(\star f^*(T\mathcal{P})) = \bigcup_{x \in \mathcal{O}} A_x(\star f^*(T\mathcal{P})) \subset \star f^*(T\mathcal{P})$ with the subspace topology. In general, $A(\star f^*(T\mathcal{P}))$ will not be an orbifold. Recall that the set of admissible *tangent* vectors at z, $A_z(\mathcal{P})$ as defined in [8] are obtained from definition 38 by replacing $\star f^*(T\mathcal{P})$ by $T\mathcal{P}$.

7. THE EXPONENTIAL MAP AND PROOF OF THEOREM 1

In this section we will need several facts about Riemannian orbifolds and the exponential map. Our reference for this material will be [8]. Throughout this section, we assume that $\mathcal{O}, \mathcal{P}$ are C^∞ smooth orbifolds and that $\mathcal{O}$ is compact (without boundary). We may assume, by [8, proposition 6.4], that both $\mathcal{O}$ and $\mathcal{P}$ are equipped with C^∞ Riemannian metrics.

The exponential map. Recall the construction of the exponential map for a smooth Riemannian orbifold $\mathcal{P}$ [8, section 6]. Assume that the collection $\{V_\alpha\}$ is a locally finite open covering of $\mathcal{P}$ by orbifold charts that are relatively compact. Let $TV_\alpha \cong (\tilde{V}_\alpha \times \mathbb{R}^m)/\Gamma_\alpha$ be a local trivialization of the tangent bundle over V_α. Denote the Riemannian exponential map on $T\tilde{V}_\alpha$ by $\widetilde{\exp}_{\tilde{V}_\alpha} : T\tilde{V}_\alpha \to \tilde{V}_\alpha$. Thus, for $\tilde{z} \in \tilde{V}_\alpha$ and $\tilde{v} \in T_{\tilde{z}}\tilde{V}_\alpha$ we have $\widetilde{\exp}_{\tilde{V}_\alpha}(\tilde{z}, t\tilde{v}) = \tilde{c}_{\tilde{z}, \tilde{v}}(t)$ where $\tilde{c}_{\tilde{z}, \tilde{v}}$ is the unit speed geodesic in $\tilde{V}_\alpha$ which starts at $\tilde{z}$ and has initial velocity $\tilde{v}$. Recall that there is an open neighborhood $\tilde{\Omega}_{\tilde{V}_\alpha} \subset T\tilde{V}_\alpha$ of the 0-section of $T\tilde{V}_\alpha$ such that $\tilde{c}_{\tilde{z}, \tilde{v}}(1)$ is defined for $\tilde{v} \in T_{\tilde{z}}\tilde{V}_\alpha \cap \tilde{\Omega}_{\tilde{V}_\alpha}$. Furthermore, by shrinking $\tilde{\Omega}_{\tilde{V}_\alpha}$ if necessary, we may assume that on $T_{\tilde{z}}\tilde{V}_\alpha \cap \tilde{\Omega}_{\tilde{V}_\alpha}$, $\widetilde{\exp}_{\tilde{V}_\alpha}(\tilde{z}, \cdot)$ is a local diffeomorphism onto a neighborhood of $\tilde{z} \in \tilde{V}_\alpha$ for each $\tilde{z} \in \tilde{V}_\alpha$. Let $\Omega_\alpha = \Pi_\alpha(\tilde{\Omega}_{\tilde{V}_\alpha})$, an open subset of $T\mathcal{P}$, and define $\Omega = \bigcup_\alpha \Omega_\alpha$. Ω is an open neighborhood of the 0-orbisection of $T\mathcal{P}$.

Definition 39. Let $z \in V_\alpha$, and $(z, v) \in \Omega_\alpha$. Choose $(\tilde{z}, \tilde{v}) \in \Pi_\alpha^{-1}(z, v)$. Then the *Riemannian exponential map* $\exp : \Omega \subset T\mathcal{P} \to \mathcal{P}$ is defined by $\exp(z, v) = \pi_\alpha \circ \widetilde{\exp}_{\tilde{V}_\alpha}(\tilde{z}, \tilde{v})$.

By [8, proposition 6.7], this exponential map is well-defined and $\widetilde{\exp}_{\tilde{V}_\alpha}$ satisfies, for all $\delta \in \Gamma_\alpha$, the equivariance relation:

$$\widetilde{\exp}_{\tilde{V}_\alpha}[\delta \cdot (\tilde{z}, \tilde{v})] = \delta \cdot \widetilde{\exp}_{\tilde{V}_\alpha}(\tilde{z}, \tilde{v}).$$

As usual we denote by $\exp_z$ the restriction of $\exp$ to a single tangent cone $T_z\mathcal{P}$. We let $B(x, r)$ denote the metric r-ball centered at z and use tildes to denote corresponding points in local coverings.

The relation between orbisections, the exponential map and complete orbifold maps. The composition of the exponential map with an orbisection of the pullback tangent orbibundle via a complete orbifold map $_\star f$ turns out to be a smooth complete orbifold map with the same equivariance relation as $_\star f$.

Proposition 40. *Let $\mathcal{O}, \mathcal{P}$ be smooth Riemannian orbifolds and let $_\star f \in C^r_{\star\mathrm{Orb}}(\mathcal{O}, \mathcal{P})$. Let σ be a C^r orbisection of the pullback tangent orbibundle $_\star f^*(T\mathcal{P})$ with $F \circ \sigma(x) \in \Omega$. Then the map $E^\sigma(x) = (\exp \circ F \circ \sigma)(x) : \mathcal{O} \to \mathcal{P}$ is a complete C^r orbifold map with representation $_\star E^\sigma = (E^\sigma, \{\tilde{E}^\sigma_x\}, \{\Theta_{E^\sigma, x}\})$ where $\tilde{E}^\sigma_x = \widetilde{\exp}_{V_{f(x)}} \circ \tilde{F}_x \circ \tilde{\sigma}_x$ and $\Theta_{E^\sigma, x} = \Theta_{f, x}$ for all $x \in \mathcal{O}$. In particular, $_\star E^\sigma \in C^r_{\star\mathrm{Orb}}(\mathcal{O}, \mathcal{P})$.*

Proof. Let $(\tilde{U}_x, \Gamma_x)$ be an orbifold chart at $x \in \mathcal{O}$. For $y \in U_x$, $\sigma(y) = (y, \xi(y)) \in {}_\star f^*(TV_{f(x)})$. Let $\tilde{\sigma}_x = (\tilde{y}, \tilde{\xi}(\tilde{y}))$ be a lift of σ and $\tilde{F}_x$ the map defined after the diagram of definition 31. Then the map $\tilde{E}^\sigma_x = \widetilde{\exp}_{V_{f(x)}} \circ \tilde{F}_x \circ \tilde{\sigma}_x$ is a C^r lift of E^σ and using the equivariance relations for σ_x, $\tilde{F}_x$ and $\widetilde{\exp}_{V_{f(x)}}$ we have for all $\gamma \in \Gamma_x$:

$$\begin{aligned}
\tilde{E}^\sigma_x(\gamma \cdot \tilde{y}) &= \widetilde{\exp}_{\tilde{V}_{f(x)}} \circ \tilde{F}_x \circ \tilde{\sigma}_x(\gamma \cdot \tilde{y}) \\
&= \widetilde{\exp}_{\tilde{V}_{f(x)}} \circ \tilde{F}_x(\gamma \cdot \tilde{\sigma}_x(\tilde{y})) \\
&= \widetilde{\exp}_{\tilde{V}_{f(x)}}(\Theta_{f,x}(\gamma) \cdot \tilde{\xi}(\tilde{y})) \\
&= \Theta_{f,x}(\gamma) \cdot \widetilde{\exp}_{\tilde{V}_{f(x)}}(\tilde{\xi}(\tilde{y})) \\
&= \Theta_{f,x}(\gamma) \cdot \widetilde{\exp}_{\tilde{V}_{f(x)}} \circ \tilde{F}_x \circ \tilde{\sigma}_x(\tilde{y}) \\
&= \Theta_{f,x}(\gamma) \cdot \tilde{E}^\sigma_x(\tilde{y}).
\end{aligned}$$

$\square$

The local manifold structure. Let $\mathcal{O}, \mathcal{P}$ and $_\star f$ be as in proposition 40. Denote by $\mathbf{0} : \mathcal{O} \to {}_\star f^*(T\mathcal{P})$, $\mathbf{0}(x) = 0_x \in {}_\star f^*(T\mathcal{P})_x$, the 0-orbisection of $_\star f^*(T\mathcal{P})$. Then $E^{\mathbf{0}}(x) = (\exp \circ F \circ \mathbf{0})(x) = f(x)$. We let $\mathcal{B}^r_{\star f}(\sigma, \varepsilon) = \mathcal{N}^r(\sigma, \varepsilon) \cap \mathcal{D}^r_{\mathrm{Orb}}({}_\star f^*(T\mathcal{P}))$. That is, $\mathcal{B}^r_{\star f}(\sigma, \varepsilon)$ is the set of C^r orbisections ε-close to σ in the C^r topology on $C^r_{\mathrm{Orb}}(\mathcal{O}, {}_\star f^*(T\mathcal{P}))$. Proposition 40 and definition 21 immediately yield the following

Proposition 41. *Let $\mathcal{O}, \mathcal{P}$ and $_\star f$ be as in proposition 40 with $\mathcal{O}$ compact. There exists $\varepsilon > 0$ and continuous map $E : \mathcal{B}^r_{\star f}(\mathbf{0}, \varepsilon) \to \mathcal{N}^r(_\star f, \varepsilon)$ defined by $E(\sigma) = E^\sigma$.*

Since $\mathcal{B}^r_{\star f}(\mathbf{0}, \varepsilon)$ is an open subset of a Banach/Fréchet space by proposition 37, the proof theorem 1 will be complete if E is shown to be a homeomorphism. We first show that E is injective.

Proposition 42. *The map $E : \mathcal{B}^r_{\star f}(\mathbf{0}, \varepsilon) \to \mathcal{N}^r(_\star f, \varepsilon)$ is injective.*

Proof. Suppose $E(\sigma) = E(\tau)$ for $\sigma, \tau \in \mathcal{B}^r_{\star f}(\mathbf{0}, \varepsilon)$. Since these are to be considered equal as complete orbifold maps, in each orbifold chart $(\tilde{U}_x, \Gamma_x)$, we must have equal local lifts: $\widetilde{\exp}_{\tilde{V}_{f(x)}} \circ \tilde{F}_x \circ \tilde{\sigma}_x(\tilde{y}) = \widetilde{\exp}_{\tilde{V}_{f(x)}} \circ \tilde{F}_x \circ \tilde{\tau}_x(\tilde{y})$. If we write in local coordinates $\tilde{\sigma}_x(\tilde{y}) = (\tilde{y}, \tilde{\xi}(\tilde{y}))$ and $\tilde{\tau}_x(\tilde{y}) = (\tilde{y}, \tilde{\eta}(\tilde{y}))$ where $\xi(\tilde{y}) = [\tilde{f}_x(\tilde{y}), v_{\tilde{f}_x(\tilde{y})}]$ and

$\eta(\tilde{y}) = [\tilde{f}_x(\tilde{y}), w_{\tilde{f}_x(\tilde{y})}]$ then

$$\widetilde{\exp}_{\tilde{V}_{f(x)}} \circ \tilde{F}_x \circ \tilde{\sigma}_x(\tilde{y}) = \widetilde{\exp}_{\tilde{V}_{f(x)}} \circ \tilde{F}_x \circ \tilde{\tau}_x(\tilde{y}) \Longleftrightarrow$$

$$\widetilde{\exp}_{\tilde{V}_{f(x)}} \circ \tilde{F}_x(\tilde{y}, \tilde{\xi}(\tilde{y})) = \widetilde{\exp}_{\tilde{V}_{f(x)}} \circ \tilde{F}_x(\tilde{y}, \tilde{\eta}(\tilde{y})) \Longleftrightarrow$$

$$\widetilde{\exp}_{\tilde{V}_{f(x)}} [\tilde{f}_x(\tilde{y}), v_{\tilde{f}_x(\tilde{y})}] = \widetilde{\exp}_{\tilde{V}_{f(x)}} [\tilde{f}_x(\tilde{y}), w_{\tilde{f}_x(\tilde{y})}]$$

Since $\widetilde{\exp}_{\tilde{V}_{f(x)}}(\tilde{f}_x(\tilde{y}), \cdot)$ is a local C^r diffeomorphism we must have $v_{\tilde{f}_x(\tilde{y})} = w_{\tilde{f}_x(\tilde{y})}$. Hence $\sigma = \tau$ (as orbifold maps) and E is injective. $\qquad\square$

The proof of the following proposition is a slightly modified version of [8, proposition 7.3].

Proposition 43. *The map $E : \mathcal{B}^r_{\star f}(\mathbf{0}, \varepsilon) \to \mathcal{N}^r(\star f, \varepsilon)$ is surjective.*

Proof. Let $\star g = (g, \{\tilde{g}_x\}, \{\Theta_{g,x}\}) \in \mathcal{N}^r(\star f, \varepsilon)$. Let $\{C_i\}$ be a finite covering of $\mathcal{O}$ by compact sets such that C_i is an orbifold chart and $g(C_i) \subset V_i$ where V_i is a relatively compact orbifold chart of $\mathcal{P}$. Let $x \in C_i$, and $\tilde{U}_x \subset \text{int}\,\tilde{C}_i$ an orbifold chart at x where the local lift $\tilde{g}_x$ to $\tilde{U}_x$ is C^0 ε-close to the local lift $\tilde{f}_x$. By lemma 24 and its proof we have $\Theta_{f,x} = \theta_{g(x)f(x)} \circ \Theta_{g,x} : \Gamma_x \to \Gamma_{f(x)}$. In particular, the action of $\Theta_{f,x}$ is the same as action $\Theta_{g,x}$ on the image $\tilde{g}_x(\tilde{U}_x) \subset \tilde{V}_{f(x)}$.

We wish to define a C^r orbisection σ so that $E(\sigma) = \star g$. We do this by defining appropriate local lifts $\tilde{\sigma}_x$. In particular, let

$$\tilde{\sigma}_x(\tilde{y}) = (\tilde{y}, \tilde{\xi}(\tilde{y})) = \left(\tilde{y}, \left[\tilde{f}_x(\tilde{y}), \widetilde{\exp}^{-1}_{\tilde{V}_{f(x)}, \tilde{f}_x(\tilde{y})} (\tilde{g}_x(\tilde{y})) \right] \right) \in {}_\star f^*(T\tilde{V}_{f(x)}).$$

With this definition, we see that

$$\tilde{E}^\sigma_x(\tilde{y}) = \widetilde{\exp}_{\tilde{V}_{f(x)}} \circ \tilde{F}_x \circ \tilde{\sigma}_x(\tilde{y})$$

$$= \widetilde{\exp}_{\tilde{V}_{f(x)}} \circ \widetilde{\exp}^{-1}_{\tilde{V}_{f(x)}, \tilde{f}_x(\tilde{y})} (\tilde{g}_x(\tilde{y})) = \tilde{g}_x(\tilde{y}).$$

This shows that $E(\sigma) = \star g$. All that remains to show is that $\tilde{\sigma}_x$ satisfies the correct equivariance relation for an orbisection. Before we do that, observe that, in general, for $\delta \in \Gamma_{f(x)}$ we have (essentially for any exponential map)

$$\widetilde{\exp}^{-1}_{\tilde{V}_{f(x)}, \tilde{f}_x(\tilde{y})} (\delta \cdot \tilde{z}) = (d\delta)_{\delta^{-1} \tilde{f}_x(\tilde{y})} \circ \widetilde{\exp}^{-1}_{\tilde{V}_{f(x)}, \delta^{-1} \tilde{f}_x(\tilde{y})} (\tilde{z})$$

$$= \delta \cdot \widetilde{\exp}^{-1}_{\tilde{V}_{f(x)}, \delta^{-1} \tilde{f}_x(\tilde{y})} (\tilde{z}).$$

Thus,

$$\tilde{\sigma}_x(\gamma \cdot \tilde{y}) = (\gamma \cdot \tilde{y}, \tilde{\xi}(\gamma \cdot \tilde{y})) = \left(\gamma \cdot \tilde{y}, \left[\tilde{f}_x(\gamma \cdot \tilde{y}), \widetilde{\exp}^{-1}_{\tilde{V}_{f(x)}, \tilde{f}_x(\gamma \cdot \tilde{y})} (\tilde{g}_x(\gamma \cdot \tilde{y})) \right] \right)$$

$$= \left(\gamma \cdot \tilde{y}, \left[\Theta_{f,x}(\gamma) \cdot \tilde{f}_x(\tilde{y}), \widetilde{\exp}^{-1}_{\tilde{V}_{f(x)}, \Theta_{f,x}(\gamma) \tilde{f}_x(\tilde{y})} (\Theta_{g,x}(\gamma) \cdot \tilde{g}_x(\tilde{y})) \right] \right)$$

$$= \left(\gamma \cdot \tilde{y}, \left[\Theta_{f,x}(\gamma) \cdot \tilde{f}_x(\tilde{y}), \Theta_{f,x}(\gamma) \cdot \widetilde{\exp}^{-1}_{\tilde{V}_{f(x)}, (\Theta_{f,x}(\gamma))^{-1} \Theta_{f,x}(\gamma) \tilde{f}_x(\tilde{y})} (\tilde{g}_x(\tilde{y})) \right] \right)$$

$$= \left(\gamma \cdot \tilde{y}, \left[\Theta_{f,x}(\gamma) \cdot \tilde{f}_x(\tilde{y}), \Theta_{f,x}(\gamma) \cdot \widetilde{\exp}^{-1}_{\tilde{V}_{f(x)}, \tilde{f}_x(\tilde{y})} (\tilde{g}_x(\tilde{y})) \right] \right)$$

$$= \left(\gamma \cdot \tilde{y}, \Theta_{f,x}(\gamma) \cdot \tilde{\xi}(\tilde{y}) \right)$$

$$= \gamma \cdot \tilde{\sigma}_x(\tilde{y})$$

which is the correct equivariance relation for an orbisection. As a result we see that the map $\sigma(x) = \Pi_x \circ \tilde{\sigma}_x(\tilde{x})$ defines a C^r orbisection of $_\star f^*(T\mathcal{P})$ and that $E(\sigma) = {}_\star g$. $\qquad\square$

The following proposition is the last step to complete the proof of the theorem 1. It gives a C^0 manifold structure to $C^r_{\star\mathrm{Orb}}(\mathcal{O}, \mathcal{P})$ where the model space for a neighborhood of $_\star f$ is the topological vector space of C^r orbisections of a pullback tangent orbibundle of $\mathcal{P}$ via $_\star f$ with the C^r topology.

Proposition 44. *The map $E : \mathcal{B}^r_{\star f}(\mathbf{0}, \varepsilon) \to \mathcal{N}^r({}_\star f, \varepsilon)$ is a homeomorphism.*

Proof. Propositions 42 and 43 show that E is bijective. Continuity of E follows from the formula for a local lift of E given in proposition 40 and continuity of E^{-1} follows from the formula for $\tilde{\sigma}_x$ given in the proof of proposition 43. $\qquad\square$

8. Building Stratified Neighborhoods

Our first task of this section will be to prove corollary 4. Let $f \in C^r_{\mathrm{Orb}}(\mathcal{O}, \mathcal{P})$. From the observation following definition 21, we have that

$$q^{-1}(\mathcal{N}^r(f, \varepsilon)) = \mathcal{N}^r({}_\star f_1, \varepsilon) \cup \cdots \cup \mathcal{N}^r({}_\star f_k, \varepsilon)$$

is a disjoint union of neighborhoods where each complete map $_\star f_i = (f, \{\tilde{f}_x\}, \{\Theta_{f,x}\}_i)$. We first partition the neighborhood $\mathcal{N}^r(f, \varepsilon)$. For each $g \in \mathcal{N}^r(f, \varepsilon)$, define $J_g \subset \{1, \ldots, k\}$ to be the set of indices j such that $q^{-1}(g) \cap \mathcal{N}^r({}_\star f_j, \varepsilon) \neq \emptyset$. For $J \subset \{1, \ldots, k\}$, define

$$\mathcal{X}_J = \{g \in \mathcal{N}^r(f, \varepsilon) \mid J = J_g\}.$$

This is a partition of $\mathcal{N}^r(f, \varepsilon)$. Of course, the partial ordering is from set inclusion: $J' \prec J \Leftrightarrow J' \subset J$. We now verify conditions (1) and (2) of definition 27 in the next two lemmas.

Lemma 45. *Each $\mathcal{X}_J$ is a submanifold of $\mathcal{N}^r(f, \varepsilon)$.*

Proof. Let $J = \{j_1, \ldots, j_l\}$. For any $g \in \mathcal{X}_J$, we have

$$q^{-1}(g) \cap \mathcal{N}^r({}_\star f_{j_1}, \varepsilon) \cap \cdots \cap \mathcal{N}^r({}_\star f_{j_l}, \varepsilon) \neq \emptyset.$$

By proposition 44, there exists unique orbisections $\sigma_{j_1}, \ldots, \sigma_{j_l}$ (of the respective pullback tangent orbibundles $_\star f^*_{j_i}(T\mathcal{P})$) such that $q(E(\sigma_{j_i})) = g$ for $i = 1, \ldots, l$. Let $(\tilde{U}_x, \Gamma_x)$ be a local chart about $x \in \mathcal{O}$ and let $\tilde{\sigma}_{j_i,x}$ denote the local lifts of the orbisection σ_{j_i} and let $\tilde{F}_{j_i,x} : \tilde{U}_x \times_{\tilde{V}_{f(x)}} T\tilde{V}_{f(x)} \to T\tilde{V}_{f(x)}$ denote the linear isomorphism $(\tilde{y}, \tilde{\xi}) \mapsto \tilde{\xi}$ given in definition 31 of the pullback tangent orbibundle. Since $\widetilde{\exp}_{\tilde{V}_{f(x)}} \circ \tilde{F}_{j_i,x} \circ \tilde{\sigma}_{j_i,x}(\tilde{y}) = \tilde{g}_x(\tilde{y})$ for all $i = 1, \ldots, l$ and since $\widetilde{\exp}_{\tilde{V}_{f(x)}}(\tilde{f}_x(\tilde{y}), \cdot)$ is a local diffeomorphism, we must have

$$\tilde{F}_{j_1,x} \circ \tilde{\sigma}_{j_1,x}(\tilde{y}) = \cdots = \tilde{F}_{j_l,x} \circ \tilde{\sigma}_{j_l,x}(\tilde{y})$$

for all $\tilde{y} \in \tilde{U}_x$. Because $\tilde{F}_{j_i,x}$ is a linear isomorphism, this relation is preserved under addition and scalar multiplication of local lifts of orbisections $\sigma_{j_i,x}$. From the proof of proposition 36, this relation descends to

$$(*) \qquad\qquad F_{j_1} \circ \sigma_{j_1}(y) = \cdots = F_{j_l} \circ \sigma_{j_l}(y)$$

for $y \in U_x$. Since F_{j_i} is a linear isomorphism when restricted to the vector space of admissible vectors $A_x({}_\star f^*_{j_i}(T\mathcal{P}))$, the set of orbisections satisfying these relations is a linear submanifold of each $\mathcal{B}_{\star f_{j_i}}(\mathbf{0}, \varepsilon)$. From this we may conclude that each

$g \in \mathcal{X}_J$ has a neighborhood modeled on a linear submanifold of $\mathcal{B}_{\star f_{j_i}}(\mathbf{0}, \varepsilon)$, which is enough to prove that $\mathcal{X}_J$ is a submanifold of $\mathcal{N}^r(f, \varepsilon)$. $\square$

Lemma 46. *If $\mathcal{X}_J \cap \overline{\mathcal{X}_{J'}} \neq \emptyset$, $J \neq J'$, then $J' \prec J$ and $\mathcal{X}_J \subset \overline{\mathcal{X}_{J'}}$.*

Proof. Let $J' = \{j_1, \ldots, j_l\}$. For $i = 1, \ldots, l$, suppose $\left\{\sigma_{j_i}^{(k)}\right\}_{k=1}^{\infty}$ is a sequence of orbisections which converges to $\overline{\sigma_{j_i}} \in \mathcal{D}_{\mathrm{Orb}}^r(\star f_{j_i}^*(T\mathcal{P}))$. Further suppose each $\sigma_{j_i}^{(k)}$ satisfies condition (*) of lemma 45. Then, by continuity, each $\overline{\sigma_{j_i}}$ satisfies (*) also. If we let $q(E(\overline{\sigma_{j_i}})) = \overline{g}$, and $J = J_{\overline{g}}$ we have shown that if $\mathcal{X}_J \cap \overline{\mathcal{X}_{J'}} \neq \emptyset$, $J \neq J'$, then $J' \prec J$ and $\mathcal{X}_J \subset \overline{\mathcal{X}_{J'}}$. $\square$

Theorem 1 with lemmas 45 and 46 together prove corollary 4. Finally the proof of corollary 5 follows from corollary 4 and lemmas 12 and 22. That is, $\mathcal{N}^r(_{\bullet}f, \varepsilon)$ is the quotient of the finite group $\mathcal{ID}_{\mathcal{N}}$ acting on $\mathcal{N}^r(f, \varepsilon)$. That the corresponding quotient map $q_{\bullet}$ restricts on each stratum to give a smooth orbifold chart follows from an argument almost identical to the argument in the proof of corollary 3 from section 3 that $q_{\bullet}$ defined a smooth orbifold chart.

An alternative view of the stratification. Up to this point, the notion of pullback bundle (definition 31) required the use of a *complete* orbifold map. Although not necessary for our results, we present a more global view of the stratification obtained above by defining directly an appropriate notion of pullback for an orbifold map f. We will use the setup of this section and the notation of definition 31. However, for convenience we will write a complete orbifold map $_{\star}f_i = (f, \{\tilde{f}_x\}, \{\Theta_{f,x,i}\})$. To begin, we let $f^*(T\mathcal{P})$ be the space defined by:

$$f^*(T\mathcal{P}) = \left(\bigsqcup_{i=1}^{k} {}_{\star}f_i^*(T\mathcal{P})\right)\Big/ \sim$$

where the equivalence relation $\sim$ is defined as follows: Let

$$\bigsqcup_{i=1}^{k} \left(\tilde{U}_x \times_{\tilde{V}_{f(x)}} T\tilde{V}_{f(x)}\right)_i$$

denote the disjoint union of k copies of $\tilde{U}_x \times_{\tilde{V}_{f(x)}} T\tilde{V}_{f(x)}$. Then in local bundle charts, for $(\tilde{y}_i, \tilde{\xi}_i) \in \left(\tilde{U}_x \times_{\tilde{V}_{f(x)}} T\tilde{V}_{f(x)}\right)_i$,

$$(\tilde{y}_i, \tilde{\xi}_i) \sim (\tilde{y}_j, \tilde{\xi}_j) \Longleftrightarrow$$
$$\tilde{y}_i = \tilde{y}_j \text{ and,}$$
$$\Theta_{f,x,i}(\gamma) \cdot \tilde{\xi}_i = \Theta_{f,x,j}(\gamma) \cdot \tilde{\xi}_j \text{ for all } \gamma \in \Gamma_x.$$

There is an obvious projection map onto $\mathcal{O}$ and the total space of $f^*(T\mathcal{P})$ is a bundle over $\mathcal{O}$. Note that there are standard continuous injections $\iota_i : {}_{\star}f_i^*(T\mathcal{P}) \to f^*(T\mathcal{P})$ and that the bundle maps $F_i : {}_{\star}f_i^*(T\mathcal{P}) \to T\mathcal{P}$ glue together to give a continuous bundle map $F : f^*(T\mathcal{P}) \to T\mathcal{P}$ satisfying $F \circ \iota_i = F_i$.

We also define for $J \subset \{1, \ldots, k\}$ the suborbifold $f^*(T\mathcal{P})_J$ of $f^*(T\mathcal{P})$ by

$$f^*(T\mathcal{P})_J = \bigsqcup_{i \in J} {}_{\star}f_i^*(T\mathcal{P}_J)/ \sim$$

where $T\mathcal{P}_J$ is the subspace of $T\mathcal{P}$ covered in bundle charts by

$$(T\tilde{V}_{f(x)})_J = \left\{ (\tilde{f}_x(\tilde{y}), \tilde{\xi}) \in T\tilde{V}_{f(x)} \mid \Theta_{f,x,i}(\gamma) \cdot \tilde{\xi} = \Theta_{f,x,j}(\gamma) \cdot \tilde{\xi} \text{ for all } i, j \in J \right\}.$$

Finally, let

$$\mathcal{D}^r_{\mathrm{Orb}}(_\star f_i^*(T\mathcal{P}))_J =$$
$$\{\sigma \in \mathcal{D}^r_{\mathrm{Orb}}(_\star f_i^*(T\mathcal{P})) \mid \tilde{F}_{x,i} \circ \tilde{\sigma}_x(\tilde{y}) \in (T\tilde{V}_{f(x)})_J \text{ for all } \tilde{y} \in \tilde{U}_x \text{ and } x \in \mathcal{O}\}.$$

Note that $\tilde{F}_{x,i} \circ \tilde{F}_{x,j}^{-1} : \mathcal{D}^r_{\mathrm{Orb}}(_\star f_j^*(T\mathcal{P}))_J \to \mathcal{D}^r_{\mathrm{Orb}}(_\star f_i^*(T\mathcal{P}))_J$ is a linear isomorphism of Banach (r finite)/Fréchet ($r = \infty$) subspaces for all $i, j \in J$ and $0 \leq r \leq \infty$. By abuse of notation, we write $\mathcal{D}^r_{\mathrm{Orb}}(f^*(T\mathcal{P}))$ for the space of C^r orbisections $\sigma : \mathcal{O} \to f^*(T\mathcal{P})$ equipped with the C^r topology. From the construction of $f^*(T\mathcal{P})$ it is clear that the Riemannian exponential map on $T\mathcal{P}$ induces a map $E : \mathcal{D}^r_{\mathrm{Orb}}(f^*T\mathcal{P}) \to C^r_{\mathrm{Orb}}(\mathcal{O}, \mathcal{P})$ as in proposition 41. For $f \in C^r_{\mathrm{Orb}}(\mathcal{O}, \mathcal{P})$ and $_\star f_i \in C^r_{\star\mathrm{Orb}}(\mathcal{O}, \mathcal{P})$ mapping to f we let $\Theta(f)_x = \{\Theta_{f,x,i}\}$ where $_\star f_i = (f, \{\tilde{f}_x\}, \{\Theta_{f,x,i}\})$.

Lemma 47. *There are neighborhoods $\mathcal{B}^r_f(\mathbf{0}, \varepsilon)$ of $\mathbf{0} \in \mathcal{D}^r_{\mathrm{Orb}}(f^*T\mathcal{P})$ and $\mathcal{N}^r(f, \varepsilon)$ of f in $C^r_{\mathrm{Orb}}(\mathcal{O}, \mathcal{P})$ so that $E : \mathcal{B}^r_f(\mathbf{0}, \varepsilon) \to \mathcal{N}^r(f, \varepsilon)$ is a homeomorphism.*

Proof. The proof follows from observing that $\{g \in C^r_{\mathrm{Orb}}(\mathcal{O}, \mathcal{P}) \mid \Theta(g)_x \subset \Theta(f)_x$ for all $x \in \mathcal{O}\}$ is an open subset (since the homomorphisms $\Theta_{f,x}$ are locally constant). By theorem 1, there is a neighborhood of each $_\star f_i$ for which the map E of proposition 41 is a homeomorphism. By taking $g \in C^r_{\mathrm{Orb}}(\mathcal{O}, \mathcal{P})$ as above and sufficiently C^r close to f, all of its preimages $_\star g_j = (g, \{\tilde{g}_x\}, \{\Theta_{g,x,j}\})$ will lie in such neighborhoods. $\square$

9. SOME INFINITE-DIMENSIONAL ANALYSIS

In this section we recall the results of global analysis that we need in order to substantiate our various claims of smoothness. For spaces of maps of finite order differentiability, a strong argument can be made that the Lipschitz categories Lip^r are better suited to questions of calculus than the more common C^r category [16]. For our purposes, however, we have chosen to use the C^r category for spaces of maps of finite order differentiability, and for spaces of maps of infinite order differentiability, we use the convenient calculus as detailed in the monographs [16, 22].

Review of the convenient calculus. For any locally convex topological vector space E the notion of a smooth curve $c \in C^\infty(\mathbb{R}, E)$ makes sense using the usual difference quotient and iterating. A mapping $f : E \to F$ between locally convex vector spaces is called *smooth* if it maps smooth curves to smooth curves. That is, if $f \circ c \in C^\infty(\mathbb{R}, F)$ for all $c \in C^\infty(\mathbb{R}, E)$. For E, F finite dimensional, or even Banach spaces, this yields the usual (Fréchet differentiable) notion of (C^∞-) smoothness [17, 21]. Unfortunately, such a simple characterization does not detect finite order (C^r-) differentiability [4]. Generalizing the fact that a map f between finite dimensional vector spaces is smooth if and only if its component functions are smooth, Frölicher, Kriegl and Michor [16, 22] introduce the notion of a *convenient* vector space: A locally convex vector space is *convenient* if every scalarwise smooth curve $c : \mathbb{R} \to E$ is smooth. c is a scalarwise smooth curve if $\ell \circ c : \mathbb{R} \to \mathbb{R}$ is smooth for all continuous linear functionals ℓ on E. For our purposes, we remark that if E is a Fréchet space then E is convenient and the locally convex topology agrees with

the c^∞-topology or final topology with respect to the set of mappings $C^\infty(\mathbb{R}, E)$. It then follows from these definitions that smooth mappings between Fréchet spaces are continuous. At this point, one can work on c^∞-open subsets of convenient vector spaces and introduce the notions of smooth manifold, smooth tangent bundle and smooth Lie group modeled on convenient vector spaces in a straightforward way.

An important feature of the convenient setting is the following theorem on Cartesian closedness:

Theorem 48 ([22, Theorem 3.12]). *Let $A_i \subset E_i$ be c^∞-open subsets in locally convex spaces, which need not be c^∞-complete. Then a mapping $f : A_1 \times A_2 \to F$ is smooth if and only if the canonically associated mapping $f^\vee : A_1 \to C^\infty(A_2, F)$ exists and is smooth.*

To substantiate the smoothness claims of Theorem 1, we use the convenient calculus in the case $r = \infty$. In the case that r is finite, we use smooth approximations and the Omega lemma 53. This will complete the proof. Throughout the remainder, we assume, as in section 7, that all orbifolds are C^∞ smooth with C^∞ Riemannian metric. Further, the orbifold $\mathcal{O}$ will be compact (without boundary).

Proposition 49. *Let $\mathcal{O}, \mathcal{P}$ be C^∞ Riemannian orbifolds.*
(1) *$C^\infty_{\star\mathrm{Orb}}(\mathcal{O}, \mathcal{P})$ has the structure of a (C^∞) smooth convenient Fréchet manifold.*
(2) *For $1 \le r < \infty$, $C^r_{\star\mathrm{Orb}}(\mathcal{O}, \mathcal{P})$ has the structure of a smooth C^∞ Banach manifold.*

Proof. We have already shown in section 7 that, for $1 \le r \le \infty$, $C^r_{\star\mathrm{Orb}}(\mathcal{O}, \mathcal{P})$ has the required structure as a topological Banach (r finite)/Fréchet ($r = \infty$) manifold. Let $_\star f \in C^r_{\star\mathrm{Orb}}(\mathcal{O}, \mathcal{P})$ and let $E_{\star f} : \mathcal{B}^r_{\star f}(\mathbf{0}, \varepsilon') \to \mathcal{N}^r(_\star f, \varepsilon')$ be a manifold chart about $_\star f$ given by proposition 44. Let $_\star g \in \mathcal{N}^r(_\star f, \varepsilon')$ and choose $0 < \varepsilon < \varepsilon'$ so that the manifold chart $E_{\star g} : \mathcal{B}^r_{\star g}(\mathbf{0}, \varepsilon) \to \mathcal{N}^r(_\star g, \varepsilon)$ is contained entirely within $\mathcal{N}^r(_\star f, \varepsilon')$. Then the chart transition map

$$E^{-1}_{\star f} \circ E_{\star g} : \mathcal{B}^r_{\star g}(\mathbf{0}, \varepsilon) \to \left(E^{-1}_{\star f} \circ E_{\star g} \right) \left(\mathcal{B}^r_{\star g}(\mathbf{0}, \varepsilon) \right) \subset \mathcal{B}^r_{\star f}(\mathbf{0}, \varepsilon')$$

is a homeomorphism between open subsets of Banach/Fréchet spaces.

We now prove (1), that is, the case where $r = \infty$: Using the convenient calculus, to show smoothness, we need to show that $E^{-1}_{\star f} \circ E_{\star g}$ takes smooth curves to smooth curves. So, let $\sigma^t : \mathbb{R} \to \mathcal{B}^\infty_{\star g}(\mathbf{0}, \varepsilon) \subset C^\infty_{\star\mathrm{Orb}}(\mathcal{O}, {}_\star g^*(T\mathcal{P}))$ be a smooth curve and let $\tilde{\sigma}^t_x : (0, 1) \to \mathcal{D}^\infty({}_\star g^*(T\tilde{V}_{g(x)})) \subset C^\infty(\tilde{U}_x, \tilde{U}_x \times_{\tilde{V}_{g(x)}} T\tilde{V}_{g(x)})$ be a smooth local equivariant lift over an orbifold chart U_x. The interval $(0, 1)$ is being chosen for convenience to make clear that we want the image of the lift $\tilde{\sigma}^t_x$ to lie in a single trivializing bundle chart. The key observation is that the computations of difference quotients for σ^t are identical in the local lift $\tilde{\sigma}^t_x$ since an orbisection must take values in the admissible bundle $A({}_\star g^*(T\mathcal{P}))$ whose fibers are the vector spaces fixed by the action of the local isotropy subgroups Γ_x (section 6). In particular, it follows that σ^t is smooth if and only if each local lift $\tilde{\sigma}^t_x$ is smooth. Using theorem 48, it follows that $\tilde{\sigma}^\wedge_x(t, \tilde{y}) : (0, 1) \times \tilde{U}_x \to \tilde{U}_x \times_{\tilde{V}_{g(x)}} T\tilde{V}_{g(x)}$ is smooth. From the formulas in section 7 for E and its local lifts and using the (C^∞-) smoothness of the Riemannian exponential map, it follows that the map $\tilde{\eta}^\wedge_x(t, \tilde{y}) = \tilde{E}^{-1}_{\star f, x} \circ \tilde{E}_{\star g, x} \circ \tilde{\sigma}^\wedge_x(t, \tilde{y}) : (0, 1) \times \tilde{U}_x \to \tilde{U}_x \times_{\tilde{V}_{f(x)}} T\tilde{V}_{f(x)}$ is smooth. Another application of theorem 48 implies

that $\tilde{\eta}_x^t = \tilde{E}_{\star f,x}^{-1} \circ \tilde{E}_{\star g,x} \circ \tilde{\sigma}_x^t : (0,1) \to \mathcal{D}^\infty(_\star f^*(T\tilde{V}_{f(x)})) \subset C^\infty(\tilde{U}_x, \tilde{U}_x \times_{\tilde{V}_{f(x)}}$
$T\tilde{V}_{f(x)})$ is smooth. Thus, by our earlier observation, $\eta^t = \left(E_{\star f}^{-1} \circ E_{\star g}\right)(\sigma^t) : \mathbb{R} \to$
$\mathcal{B}_{\star f}^\infty(\mathbf{0}, \varepsilon') \subset C_{\star \mathrm{Orb}}^r(\mathcal{O}, _\star f^*(T\mathcal{P}))$ is a smooth curve. This completes the proof of
(1).

We now prove (2), which is the case when r is finite. We are grateful to the referee for reminding us of the point that, in the case of finite r, the argument above cannot show that transition maps are C^∞ since the pullback tangent orbibundle $_\star f^*(T\mathcal{P})$ is only as smooth as $_\star f$ is. To get around this difficulty, we use C^∞ approximations. The essential idea is that since the C^∞ orbifold maps are dense in the C^r orbifold maps, we are able to cover $C_{\star \mathrm{Orb}}^r(\mathcal{O}, \mathcal{P})$ by (Banach) manifold charts centered at C^∞ smooth $_\star f$. In particular, let $_\star \overline{f} \in C_{\star \mathrm{Orb}}^\infty(\mathcal{O}, \mathcal{P})$ be a smooth C^∞ approximation to $_\star f$ and $_\star \overline{g} \in C_{\star \mathrm{Orb}}^\infty(\mathcal{O}, \mathcal{P})$ be a smooth C^∞ approximation to $_\star g$. Consider the corresponding C^∞ pullback orbibundles $_\star \overline{f}^*(T\mathcal{P})$ and $_\star \overline{g}^*(T\mathcal{P})$. Working locally in local orbibundle charts, let $\tilde{F}_x : \tilde{U}_x \times_{\tilde{V}_{\overline{f}(x)}} T\tilde{V}_{\overline{f}(x)} \to T\tilde{V}_{\overline{f}(x)}$ and $\tilde{G}_x : \tilde{U}_x \times_{\tilde{V}_{\overline{g}(x)}} T\tilde{V}_{\overline{g}(x)} \to T\tilde{V}_{\overline{g}(x)}$, be the equivariant maps defined in the paragraph following the commutative diagram in definition 31. Then

$$\omega = \left(\widetilde{\exp}_{\tilde{V}_{\overline{f}(x)}} \circ \tilde{F}_x\right)^{-1} \circ \left(\widetilde{\exp}_{\tilde{V}_{\overline{g}(x)}} \circ \tilde{G}_x\right) : \tilde{U}_x \times_{\tilde{V}_{\overline{g}(x)}} T\tilde{V}_{\overline{g}(x)} \to \tilde{U}_x \times_{\tilde{V}_{\overline{f}(x)}} T\tilde{V}_{\overline{f}(x)}$$

is a smooth C^∞ fiber-respecting equivariant map. Using the Omega lemma 53 and equivariance then implies that the chart transition map

$$E_{\star \overline{f}}^{-1} \circ E_{\star \overline{g}} : \mathcal{B}_{\star \overline{g}}^r(\mathbf{0}, \varepsilon) \to \left(E_{\star \overline{f}}^{-1} \circ E_{\star \overline{g}}\right)\left(\mathcal{B}_{\star \overline{g}}^r(\mathbf{0}, \varepsilon)\right) \subset \mathcal{B}_{\star \overline{f}}^r(\mathbf{0}, \varepsilon')$$

is C^∞ smooth. Thus, by choosing the smooth approximations $_\star \overline{f}$ and $_\star \overline{g}$ close enough to $_\star f$ and $_\star g$, respectively, we can produce the desired local (Banach) manifold charts with smooth C^∞ transition map. This completes the proof of (2). □

A useful consequence of the observation made in proposition 49 is the following (compare [22, Lemma 42.5]):

Corollary 50. *The following conditions on a curve $c = (f^t, \{\tilde{f}_x^t\}, \{\Theta_{f^t, x}\}) : \mathbb{R} \to C_{\star \mathrm{Orb}}^\infty(\mathcal{O}, \mathcal{P})$ are equivalent:*

> (1) *c is smooth*
> (2) *each local equivariant lift $\tilde{f}_x^t : (0,1) \to C^\infty(\tilde{U}_x, \tilde{V}_z)$ is smooth*
> (3) *each local equivariant lift $\tilde{f}^\wedge(t, x) : (0,1) \times \tilde{U}_x \to \tilde{V}_z$ is smooth*

Proof. Note that the interval $(0,1)$ is being chosen for convenience to make clear that we want the image of the lift $\tilde{f}_x^t$ to lie in a single orbifold chart. (1) $\Longleftrightarrow$ (2) follows from the observation that smoothness of c is equivalent in local charts to smoothness of a curve into a space of orbisections. This in turn is equivalent to smoothness of the local equivariant lifts of the orbisections as in proposition 49. This, of course, is equivalent to smoothness of the local equivariant lifts $\tilde{f}_x^t$. (2) $\Longleftrightarrow$ (3) follows from theorem 48. □

Lemma 51. *Let $_\star f \in C_{\star \mathrm{Orb}}^r(\mathcal{O}, \mathcal{P})$. For $1 \leq s \leq r$, let $I : \mathcal{N}^s(_\star f, \varepsilon) \to \mathcal{N}^s(I \circ _\star f, \varepsilon)$, $_\star g \mapsto I \circ _\star g$, be the homeomorphism from lemma 22. Then I is a C^∞ diffeomorphism.*

Proof. By lemma 22 and theorem 1 via proposition 49, we know that I is a map between open subsets of a smooth C^∞ Banach (r finite)/Fréchet ($r = \infty$) manifold. By [22, section 27], it suffices to show that

$$\sigma^t = E^{-1}_{I \circ {}_\star f} \circ I \circ {}_\star g^t : \mathbb{R} \to \mathcal{D}^s_{\mathrm{Orb}}((I \circ {}_\star f))^*(T\mathcal{P}))$$

is smooth for any smooth curve ${}_\star g^t = (g^t, \{\tilde{g}^t_x\}, \{\Theta_{g^t,x}\})$. From an argument similar to that given in proposition 49, it follows that each local equivariant lift $\tilde{g}^t_x : (0,1) \to C^s(\tilde{U}_x, \tilde{V}_{f(x)})$ is smooth. Using the discussion after example 9, for $I = (\mathrm{Id}, \{\eta_x \cdot \tilde{y}\})$ we have

$$I \circ {}_\star g^t = (g^t, \{\eta_x \cdot \tilde{g}^t_x\}, \{\gamma \mapsto \eta_x \Theta_{g^t,x}(\gamma)\eta_x^{-1}\}).$$

Then by the formulas in proposition 43, σ^t has local equivariant lifts

$$\tilde{\sigma}^t_x(\tilde{y}) = (\tilde{y}, \tilde{\xi}^t(\tilde{y})) =$$
$$\left(\tilde{y}, \left[\eta_x \cdot \tilde{f}_x(\tilde{y}), \widetilde{\exp}^{-1}_{\tilde{V}_{f(x)}, \eta_x \cdot \tilde{f}_x(\tilde{y})} \left(\eta_x \cdot \tilde{g}^t_x(\tilde{y}) \right) \right] \right) \in (I \circ {}_\star f)^*(T\tilde{V}_{f(x)}).$$

Since the exponential map and action of local isotropy subgroups are (C^∞-) smooth in charts and $\tilde{g}^t_x$ is smooth, we see that $\tilde{\sigma}^t_x$ is smooth. It then follows that σ^t is smooth by corollary 50 or the observation in the proof of proposition 49, and this completes the proof. $\square$

The next result we will need is that composition in our spaces of smooth orbifold maps is smooth.

Lemma 52. *Let $\mathcal{O}$, $\mathcal{P}$ and $\mathcal{R}$ be smooth C^∞ compact orbifolds without boundary. Then the composition mappings*

$$\mathrm{comp} : C^\infty_{\star\mathrm{Orb}}(\mathcal{P}, \mathcal{R}) \times C^\infty_{\star\mathrm{Orb}}(\mathcal{O}, \mathcal{P}) \to C^\infty_{\star\mathrm{Orb}}(\mathcal{O}, \mathcal{R}), \qquad ({}_\star f, {}_\star g) \mapsto {}_\star f \circ {}_\star g,$$
$$\mathrm{comp} : C^\infty_{\blacklozenge\mathrm{Orb}}(\mathcal{P}, \mathcal{R}) \times C^\infty_{\blacklozenge\mathrm{Orb}}(\mathcal{O}, \mathcal{P}) \to C^\infty_{\blacklozenge\mathrm{Orb}}(\mathcal{O}, \mathcal{R}), \qquad ({}_\blacklozenge f, {}_\blacklozenge g) \mapsto {}_\blacklozenge f \circ {}_\blacklozenge g,$$
$$\mathrm{comp} : C^\infty_{\mathrm{Orb}}(\mathcal{P}, \mathcal{R}) \times C^\infty_{\mathrm{Orb}}(\mathcal{O}, \mathcal{P}) \to C^\infty_{\mathrm{Orb}}(\mathcal{O}, \mathcal{R}), \qquad (f, g) \mapsto f \circ g,$$
$$\mathrm{comp} : C^\infty_{\mathrm{red}}(\mathcal{P}, \mathcal{R}) \times C^\infty_{\mathrm{red}}(\mathcal{O}, \mathcal{P}) \to C^\infty_{\mathrm{red}}(\mathcal{O}, \mathcal{R}), \qquad ({}_\bullet f, {}_\bullet g) \mapsto {}_\bullet f \circ {}_\bullet g,$$

are smooth.

Proof. By lemmas 10, 12, 22, and the observation following definition 21, it suffices to prove the result for the complete orbifold maps. We use an argument analogous to [22, Theorem 42.13]. Namely, let $(c_1, c_2) : \mathbb{R} \to C^\infty_{\star\mathrm{Orb}}(\mathcal{P}, \mathcal{R}) \times C^\infty_{\star\mathrm{Orb}}(\mathcal{O}, \mathcal{P})$ be a smooth curve. Then $(\mathrm{comp} \circ (c_1, c_2))(t)(x) = c_1^\wedge(t, c_2^\wedge(t, x))$ is smooth by corollary 50. Hence comp is smooth. $\square$

Finally, for finite order differentiability, we will need to refer to the Omega lemma [1, 24, 28] as stated in [29]:

Lemma 53 (Omega lemma). *Let M be a C^∞ compact manifold and let $\tau : E \to M$ and $\tau' : E' \to M$ be C^∞ vector bundles over M. Let $U \subset E$ be open and let $\omega : U \subset E \to E'$ be a C^∞ fiber-respecting map. Let $\mathcal{D}^r(U) = \{\xi \in \mathcal{D}^r(\tau) \mid \xi(M) \subset U\}$. Then the induced map*

$$\Omega_\omega : \mathcal{D}^r(U) \subset \mathcal{D}^r(\tau) \to \mathcal{D}^r(\tau'), \ \Omega_\omega(\xi) = \omega \circ \xi$$

is a C^∞ map. If ω is only C^{r+k}, then Ω_ω is C^k.

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